

Twisted supersymmetric gauge theories and orbifold lattices

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ABSTRACT: We examine the relation between twisted versions of the extended supersymmetric gauge theories and supersymmetric orbifold lattices. In particular, for the $\mathcal{N} = 4$ SYM in $d = 4$, we show that the continuum limit of orbifold lattice reproduces the twist introduced by Marcus, and the examples at lower dimensions are usually Blau-Thompson type. The orbifold lattice point group symmetry is a subgroup of the twisted Lorentz group, and the exact supersymmetry of the lattice is indeed the nilpotent scalar supersymmetry of the twisted versions. We also introduce twisting in terms of spin groups of finite point subgroups of R -symmetry and spacetime symmetry.

KEYWORDS: Extended Supersymmetry, Lattice Gauge Field Theories, Topological Field Theories.

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1. Introduction

This paper is devoted to the study of the relation between supersymmetric orbifold lattices and twisted versions of extended supersymmetric gauge theories. This turns out to be useful in many respects. The viewpoint of this paper explains many oddities of orbifold lattices, such as associating spinless bosons of the continuum with the link fields on the lattice, and associating double-valued spinors of the continuum with single-valued representations of the lattice point group symmetry. To the reader acquainted with the so called “topological” twisting this should all sound natural and be thought as a lattice version of it. And this is indeed true. Most of this paper is a study of representation theory of continuous and finite symmetry groups to convey this picture. Making the orbifold lattice-twisted continuum theory correspondence clear also fulfills some curiosities on the relation between the two recent independent approaches on supersymmetric lattices, and in fact it reconciles them. This will be made more precise.

The formulation of the twisted theories was initiated by Witten in his classic work on Donaldson theory of four manifolds [1]. Witten constructed a twisted version of the asymptotically free $\mathcal{N} = 2$ supersymmetric Yang-Mills theory and calculated certain topological correlators, both in the ultraviolet by taking advantage of the weak coupling limit [1], and in the infrared, long distance point of view [2]. As these correlators are metric independent, they naturally come out to be the same. The technique for constructing twisted versions of other extended supersymmetric gauge theories, such as $\mathcal{N} = 4$ SYM, has been investigated in depth [3–5] and the ones which are relevant to the discussions of orbifold lattices are due to Marcus [6], and Blau and Thompson [7].

There are two recent independent approaches for the construction of a nonperturbative regularization of the supersymmetric gauge theories, and as stated earlier, one of the primary goal of this paper is to make the relation between the two precise. The first approach, the orbifold lattice, is based on an orbifold projection of a supersymmetric matrix model [8–12]. The projection generates a lattice theory while preserving a subset of the supersymmetries of the target theory and benefits from the deconstruction limit [13]. The other approach, pioneered by Catterall, [14–18] uses twists of Witten type along with Dirac-Kähler fermions. The main idea is to express the continuum action in a twisted form and discretize the theory by keeping a subset of the nilpotent (up to gauge transformations) supersymmetries exact even at finite lattice spacing. The Dirac-Kähler fermions have a geometric realization on the lattice and are usually associated with p -cells. Sugino pursued an approach based on “balanced topological field theory form” [5] and chose to put fermions on the sites [19–22]. There are also claims that the full twisted superalgebra can be incorporated to the lattice with a modified definition of Leibniz rule [23–25]. The outcomes of these two approaches are not identical. The reader may wonder why this is so, considering that the orbifold lattices produce twisted theories in the continuum. In a nutshell, the difference between the two approaches can be traced to the non-uniqueness of the embedding of the scalar supersymmetries on the lattice. As we will see, in the twisted formulation of these theories there is usually more than one scalar supercharge and any linear combination can be used on the lattice. (Also see the references [26–32] for related works.)

One of the main outcomes of our analysis is that the discrete point group symmetry of the orbifold lattices is not a subgroup of the Lorentz group *per se*, but the twisted version of it. In this viewpoint, the scalars of the physical theory turns out to be vectors under the twisted Lorentz symmetry, which explain their appearance as the link fields. Moreover, the spinors (double valued representations) of the physical theory transform in the single-valued integer spin representations of the twisted theory. Therefore, spinors of the continuum theory are associated with single valued lattice representations. Hence, their appearances on the p -cells, sites, links, faces etc. of the hypercubic lattice can be naturally understood. We reach the same conclusions in two different ways. In the bulk of the paper, we construct twistings in terms of continuous groups, then translate the outcome to the lattice. In a short appendix, we sketch a complementary approach. We consider double-valued finite groups, which are indeed the discrete spacetime and discrete R -symmetries. Then, we show how twisting glues objects of half-integer spin into integer spin multiplets of the diagonal subgroup of discrete R and spacetime symmetry.

We also explain the relation between the A_d^* lattices and the twisted versions of $\mathcal{Q} = 16$ supercharge target theories in d dimensions. The A_d^* lattices are the most symmetric lattices, in particular they are more symmetric than the hypercubic ones. This is important when considering the quantum continuum limit and renormalization of these theories. The greater the symmetry of the spacetime lattice, the fewer relevant and marginal operators will exist. Therefore, the most symmetrical arrangements of the lattices are preferred to minimize the fine tunings in attaining the continuum limit. The point group symmetry of A_d^* lattices involves at least the permutation group S_{d+1} (not S_d as in the case of d dimensional cubic lattice). We should emphasize that the group S_{d+1} does not have double valued (spinor) representations at all, even though all the target theories possess spinor representations. We will observe that there is a close relation between the finite group S_{d+1} and continuum twisted Lorentz group and their representations. This will be discussed in depth in section 3.3 and is one of the main results of this paper.

The twisted theories emerging from the orbifold lattices are examined in the context of the topological twisting of the extended supersymmetric field theories. In four dimensions, the twist of $\mathcal{N} = 4$ is introduced by Marcus [6]. The three dimensional $\mathcal{N} = 4$ and $\mathcal{N} = 8$ and two dimensional $\mathcal{N} = (8, 8)$, $\mathcal{N} = (4, 4)$ theories are presented by Blau and Thompson [7] and are examined in more detail in [33, 34]. The twist of the two dimensional $\mathcal{N} = (2, 2)$ theory seems to be a new example of [6, 7] type and is examined in more detail here. Conversely, starting with the continuum form of the twisted theory, it is possible to reverse engineer the hypercubic orbifold lattice by using a simple recipe given by Catterall [16].

2. Maximal twisting and orbifold projection

In this section, we briefly review the twistings of extended supersymmetric gauge theories in the continuum formulation on \mathbb{R}^d [1] and sketch its relation to orbifold projections of supersymmetric matrix models. The theories of interest have a Euclidean rotation group $SO(d)_E$ and possess a global R -symmetry group G_R . For six of the theories shown in table 1, the R -symmetry group possess a $SO(d)_R$ subgroup. Hence, the full global symmetry of the supersymmetric theory has a subgroup $SO(d)_E \times SO(d)_R \subset SO(d)_E \times G_R$. To construct the twisted theory, we embed a new rotation group $SO(d)'$ into the diagonal sum of $SO(d)_E \times SO(d)_R$, and declare this $SO(d)'$ as the new Lorentz symmetry of the theory.¹

Since the details of each such construction are slightly different, let us restrict to generalities first. Let us assume that a fermionic field which is a spacetime spinor, is in spinor representation of R -symmetry group $SO(d)_R$ as well. Since the product of two half-integer spin is always an integer spin, *all* Grassmann odd degrees of freedom are in integer spin representations of $SO(d)'$. We can express the fermions as a direct sum of scalars, vectors, i.e as p -form tensors. Let us label a p -form fermion as $\psi^{(p)}$. In all of our applications, the \mathcal{Q} many fermions of a target field theory in d dimensions are distributed

¹We will not distinguish spin groups $Spin(n)$ from $SO(n)$ unless otherwise specified.

Theory	Lorentz	$\mathcal{Q} = 4$	$\mathcal{Q} = 8$	$\mathcal{Q} = 16$
$d = 2$	SO(2)	SO(2) \times U(1)	SO(4) \times SU(2)	SO(8)
$d = 3$	SO(3)	U(1)	SO(3) \times SU(2)	SO(7)
$d = 4$	SO(4)	U(1)	SO(2) \times SU(2)	SO(6)

Table 1: The R -symmetry groups of various supersymmetric gauge theories obtained by dimensionally reducing minimal $\mathcal{N} = 1$ theories from $d = 4, 6, 10$ dimensions. These R -symmetries are the product of the global symmetry due to reduced dimensions and the R -symmetry of the theory prior to reduction.

to multiplets of $SO(d)'$ as

$$\text{fermions} \rightarrow \frac{\mathcal{Q}}{2^d} (\psi^{(0)} \oplus \psi^{(1)} \oplus \dots \psi^{(d)}) \tag{2.1}$$

where the multiplicative factor up front is one, two or four. For a given p -form, there are $\frac{\mathcal{Q}}{2^d} \binom{d}{p}$ fermions. Summing over all p , we obtain the total number of fermions in the target theory: $\frac{\mathcal{Q}}{2^d} \sum_{p=0}^d \binom{d}{p} = \mathcal{Q}$

Turning to Grassmann even fields, the gauge bosons V_μ transforming as $(d, 1)$ and the spacetime scalars S_μ transforming as $(1, d)$ under the $SO(d)_E \times SO(d)_R$ level. Both transform as vectors (d) under the $SO(d)'$. If there are more than d scalars in the untwisted theory, they become either 0-forms or d -forms under $SO(d)'$.

This type of twist is sometimes referred as *maximal twist* as it involves the twisting of the full Lorentz symmetry group as opposed to twisting its subgroup. In this sense, the four dimensional $\mathcal{N} = 2$ theory can only admit a *half twisting* as its R -symmetry group is not as large as $SO(4)_E$ [1]. The other two theories, $\mathcal{N} = 1$ in $d = 4$ and $\mathcal{N} = 1$ in $d = 3$ shown in table 1 do not admit a nontrivial twisting as there is no nontrivial homomorphism from their Euclidean rotation group to their R -symmetry group.

The action expressed in terms of the representation of the twisted Lorentz group $SO(d)'$ instead of the ones of the usual Lorentz symmetry, is called twisted action. The twisted version can be expressed as a sum of Q -exact and Q -closed terms, where Q is the supersymmetry associated with scalar supersymmetry transformation. ² As it is well known, so long as the usual Lorentz symmetry is not gauged, i.e., on flat spacetime, the twisted theory is merely a rewriting of the physical theory, and indeed possess all the supersymmetries of the physical theory. ³

The main point of this twist is that none of the degrees of freedom are spinors under $SO(d)'$. Both bosons and fermions are in integer spin representations. They are p -form tensors of $SO(d)'$. This particular form of the twisted theory is the bridge to orbifold

²One can make this theory topological by interpreting the scalar supercharge Q as a BRST operator [1]. Even without doing so, one can still say that the physical theory has a set of topological observables, appropriately defined correlators of the twisted operators.

³In fact, if the base space of the theory is an arbitrary d -dimensional curved manifold M^d , then only the scalar supercharge is preserved. It is somehow peculiar that the discretized background spacetime (lattice) also respects the one and same nilpotent scalar supersymmetry.

lattices. Given such a twisted theory, it is natural to associate a p -form continuum field with a p -cell field on the hypercubic lattice. This is exactly what an orbifold lattice does. The orbifold projection places the fermions to sites, links, faces, i.e, to p -cells. This is in agreement with our expectation from the twisted rotation symmetry $SO(d)'$. On the orbifold lattice, there are also complex bosons (complexification of S_μ and V_μ as $(S_\mu \pm iV_\mu)/\sqrt{2}$) associated with oppositely oriented links and certain fields associated with p -cells. We refer the reader to ref. [10, 8] for a detailed explanation of the orbifold projection and \mathbf{r} -charge assignments. By using the analysis of ref. [10], we see that \mathbf{r} -charge assignment is intimately related to how a field transforms in the continuum. Mainly, the total number of nonzero components of the \mathbf{r} -charge is the degree p of the tensor representation of $SO(d)'$. The signs of components of \mathbf{r} determine the orientation of the corresponding lattice field. For example, on a $d = 2$ dimensional square lattice, we associate fermions with $\mathbf{r} = (0, 0)$ with 0-cell, $\mathbf{r} = (1, 0)$ with 1-cells in \mathbf{e}_1 direction, $\mathbf{r} = (0, 1)$ with 1-cells in \mathbf{e}_2 -direction and $\mathbf{r} = (-1, -1)$ with a 2-cell field in $-\mathbf{e}_1 - \mathbf{e}_2$ direction. These respectively become zero, one and two form tensor fermions as in eq. (2.1) under the continuum $SO(d)'$.

One may ask how does these orbifold projections know about the representations of the twisted group. Recall that the \mathbf{r} -charges are given in an appropriate abelian subgroup of the *full* R -symmetry group of a zero dimensional matrix model. This matrix model is obtained by dimensionally reducing the target theory to zero dimension and possesses *at least* an $SO(d)_E \times G_R$ R -symmetry group. The G_R is the R -symmetry prior to reduction and more importantly, $SO(d)_E$, which used to be the Lorentz symmetry of the target theory, is an R -symmetry of matrix theory. The full R -symmetry group of the matrix theory is in general larger than $SO(d)_E \times G_R$. For example, for $\mathcal{N} = 4$ SYM theory reduced from $d = 4$ to $d = 0$ dimensions has a manifest $SO(4)_E \times SO(6)_R$ R -symmetry, but it clearly enhances to $SO(10)_R$. The choice of \mathbf{r} -charges mixes the global Lorentz R -symmetry $SO(d)_E$ with G_R in a profound way. It is in fact a form of twisting. Let us again restrict to $\mathcal{N} = 4$ SYM theory and its reduced version. The reduced version has an $SO(10)_R$. For hypercubic lattices, the $SO(10)_R \rightarrow SO(8) \times SO(2) \rightarrow SU(4) \times U(1) \times U(1)$ branching plays a fundamental role. In fact, the \mathbf{r} charges is embedded in $U(1)^4$ subgroup of $SU(4) \times U(1) \times U(1)$ and the intact $U(1)$ remains to be an R -symmetry of lattice theory. One may wonder what does this $SU(4)$ has anything to do with the diagonal $SO(4)'$ of continuum theory. The answer is somewhat subtle. The lattice which is obtained by the orbifold projection has a finite nonabelian point group symmetry, which is the Weyl group of $SU(4)$, i.e., $Weyl(SU(4)) = S_4$ isomorphic to the permutation group S_4 . And in fact, this Weyl group is the discrete subgroup of diagonal $SO(4)'$, i.e., $S_4 \subset SO(4)'$. Under a conveniently chosen abelian subgroup of full R -symmetry, the fermions carry integer charges as bosons and they form supermultiplets. These multiplets transform in representation of non-abelian point group symmetry (or equivalently Weyl group).

The proper understanding of the more symmetric A_d^* lattices, which arise for $\mathcal{Q} = 16$ supercharge target theories in d dimensions, from the twisted supersymmetry viewpoint is a little bit more involved, but is a worthy endeavor. The A_d^* lattices are the most symmetric lattices, and the greater the symmetry of the spacetime lattice, the fewer relevant and marginal operator will exist. The point group symmetry of A_d^* involves the Weyl group of

$SU(d + 1)$, rather than $SU(d)$. For example, the highly symmetric lattice A_4^* for $\mathcal{N} = 4$ SYM theory has an $S_5 = Weyl(SU(5))$ point group symmetry, which is much larger than the point group symmetry of hypercubic lattice. The classification of the fields on the A_4^* lattice under the point group symmetry is discussed in detail in the next sections. As we will see, there is a close relation between $Weyl(SU(d + 1)) = S_{d+1}$ and continuum twisted rotation group $SO(d)'$ and their representations.

This line of reasoning teaches us that the point group symmetry of the lattice is *not* a subgroup of the Euclidean Lorentz group, but in fact a discrete subgroup of the twisted rotation group $SO(d)'$. In the continuum, the orbifold lattice theory becomes the twisted version of the desired target field theory. The change of variables which takes the twisted form to the canonical form essentially undoes the twist.

3. Marcus's twist of $\mathcal{N} = 4$ SYM in $d = 4$

There are various possible twists of the $\mathcal{N} = 4$ SYM theory in four dimensions [3, 4, 6]. The one we will consider and which emerges out of the orbifold lattice naturally is due to Marcus. Here, we briefly outline the twisting procedure. One interesting property of this twisting is that it admits a superfield formulation.

The $\mathcal{N} = 4$ SYM theory in $d = 4$ dimensions possesses a global Euclidean Lorentz symmetry $SO(4)_E \sim SU(2) \times SU(2)$, a global R -symmetry group $SO(6) \sim SU(4)$. The R -symmetry contains a subgroup $SO(4)_R \times U(1)$. To construct the twisted theory, we take the diagonal sum of $SO(4)_E \times SO(4)_R$ and declare it the new rotation group. Since the $U(1)$ part of the symmetry group is undisturbed, it remains as a global R -symmetry of the twisted theory.

Under the $G = (SU(2) \times SU(2))_E \times (SU(2) \times SU(2))_R$ symmetry, the fermions transform as $(2, 1, 2, 1) \oplus (2, 1, 1, 2) \oplus (1, 2, 1, 2) \oplus (1, 2, 2, 1)$. These fields, under $G' = SU(2)' \times SU(2)' \times U(1)$ (or under $SO(4)' \times U(1)$) transform as⁴

$$\begin{aligned} \text{fermions} &\rightarrow (1, 1)_{\frac{1}{2}} \oplus (2, 2)_{-\frac{1}{2}} \oplus [(3, 1) \oplus (1, 3)]_{\frac{1}{2}} \oplus (2, 2)_{-\frac{1}{2}} \oplus (1, 1)_{\frac{1}{2}} \\ &\rightarrow 1_{\frac{1}{2}} \oplus 4_{-\frac{1}{2}} \oplus 6_{\frac{1}{2}} \oplus 4_{-\frac{1}{2}} \oplus 1_{\frac{1}{2}}. \end{aligned} \tag{3.1}$$

The magic of this particular embedding is clear. There are two spin zero fermions, and all the fermions are now in the integer spin representation of the twisted Lorentz symmetry $SO(4)'$. They transform as scalars, vectors, and higher rank p -form tensors. We parametrize these Grassmann valued tensors, accordingly, $(\lambda, \psi^\mu, \xi_{\mu\nu}, \xi^{\mu\nu\rho}, \psi_{\mu\nu\rho\sigma})$.

The gauge boson V_μ which transform as $(2, 2, 1, 1)$ under the group G becomes $(2, 2)$ under G' . Similarly, four of the scalars S_μ transforming as $(1, 1, 2, 2)$ are elevated to the same footing as the gauge boson and transform as $(2, 2)$ under twisted rotation group. The complexification of the two vector plays a more fundamental role in the formulation. We

⁴Twice of the $U(1)$ charge is usually called the ghost number in the topological counterpart of this theory.

therefore define the complex vector fields ⁵

$$z^\mu = (S^\mu + iV^\mu)/\sqrt{2}, \quad \bar{z}_\mu = (S_\mu - iV_\mu)/\sqrt{2} \quad \mu = 1, \dots, 4 \quad (3.2)$$

Since there are two types of vector fields, there are indeed two types of complexified gauge covariant derivative appearing in the formulation. These are holomorphic and antiholomorphic covariant derivatives

$$\mathcal{D}^\mu \cdot = \partial^\mu \cdot + \sqrt{2}[z^\mu, \cdot], \quad \bar{\mathcal{D}}_\mu \cdot = -\partial_\mu \cdot + \sqrt{2}[\bar{z}_\mu, \cdot], \quad (3.3)$$

Only three combination of the covariant derivatives (similar to the F -term and D -term in the $\mathcal{N} = 1$ gauge theories) appear in the formulation. These are

$$\begin{aligned} \mathcal{F}^{\mu\nu} &= -i[\mathcal{D}^\mu, \mathcal{D}^\nu] = F_{\mu\nu} - i[S_\mu, S_\nu] - i(D_\mu S_\nu - D_\nu S_\mu) \\ \bar{\mathcal{F}}_{\mu\nu} &= -i[\bar{\mathcal{D}}_\mu, \bar{\mathcal{D}}_\nu] = F_{\mu\nu} - i[S_\mu, S_\nu] + i(D_\mu S_\nu - D_\nu S_\mu) \\ (-id) &= \frac{1}{2}[\bar{\mathcal{D}}_\mu, \mathcal{D}^\mu] + \dots = -D_\mu S_\mu + \dots \end{aligned} \quad (3.4)$$

where $D_\mu \cdot = \partial_\mu \cdot + i[V_\mu, \cdot]$ is the usual covariant derivative and $F_{\mu\nu} = -i[D_\mu, D_\nu]$ is the nonabelian field strength. The field strength $\mathcal{F}^{\mu\nu}(x)$ is holomorphic, it only depends on complexified vector field z^μ and not on \bar{z}_μ . Likewise, $\bar{\mathcal{F}}_{\mu\nu}$ is anti-holomorphic. The $(-id)$ will come out of the solutions of equations of motion for auxiliary field d and dots stands for possible scalar contributions. These combination arises from all of the orbifold lattice constructions, and is one of the reasons for considering this type of twist.

Finally, the two other scalars remains as scalars under the twisted rotation group. Since one of the scalars is the superpartner (as will be seen below) of the four form fermion, we label them as $(z_{\mu\nu\rho\sigma}, \bar{z}^{\mu\nu\rho\sigma})$. To summarize, the bosons transform under G' as

$$\text{bosons} \rightarrow z_{\mu\nu\rho\sigma} \oplus z^\mu \oplus \bar{z}_\mu \oplus \bar{z}^{\mu\nu\rho\sigma} \rightarrow [(1, 1)_1 \oplus (2, 2)_0 + (2, 2)_0 + (1, 1)_{-1}] \quad (3.5)$$

As can be seen easily from the decomposition of the fermions, there are two Lorentz singlet supercharges $(1, 1)$ under the twisted Lorentz group and either of these (or their linear combinations) can be used to write down the Lagrangian of the four dimensional theory in ‘‘topological’’ form. The difference in lattices obtained in [35, 17, 21] and orbifold lattices [8] is tightly related to the choice of the scalar supercharge, and this will be further discussed in section 3.4. Here, we use the spin zero supercharge associated with λ (motivated by the orbifold lattice). This produces the transformations given by [6].

The continuum off-shell supersymmetry transformations are given by

$$\begin{aligned} Q\lambda &= -id, & Qd &= 0 \\ Qz^\mu &= \sqrt{2} \psi^\mu, & Q\psi^\mu &= 0 \\ Q\bar{z}_\mu &= 0 \\ Q\xi_{\mu\nu} &= -i\bar{\mathcal{F}}_{\mu\nu} \end{aligned}$$

⁵Throughout this paper, $\mu, \nu, \rho, \sigma \dots$ are $SO(d)'$ or d -dimensional hypercubic indices and summed over $1, \dots, d$. The indices m, n, \dots are indices for permutation group S_{d+1} (for A_d^* lattices) and are summed over $1, \dots, (d+1)$.

$$\begin{aligned}
Q\xi^{\nu\rho\sigma} &= \sqrt{2}\bar{\mathcal{D}}_\mu\bar{z}^{\mu\nu\rho\sigma} \\
Qz_{\mu\nu\rho\sigma} &= \sqrt{2}\psi_{\mu\nu\rho\sigma}, & Q\psi_{\mu\nu\rho\sigma} &= 0 \\
Q\bar{z}^{\mu\nu\rho\sigma} &= 0
\end{aligned}
\tag{3.6}$$

where d is an auxiliary field introduced for the off-shell completion of the supersymmetry algebra. Clearly, the scalar supercharge is nilpotent

$$Q^2 \cdot = 0. \tag{3.7}$$

owing to the anti-holomorphy of $\bar{\mathcal{F}}_{\mu\nu}$ etc. The fact that the subalgebra ($Q^2 = 0$) does not produce any spacetime translations makes it possible to carry it easily onto the lattice. The exact nilpotency, as opposed to being nilpotent modulo gauge transformation has a technical advantage. It admits a rather exotic superfield formulation of the target supersymmetric field theory which will be discussed in the next section.

The twisted Lagrangian may be written as a sum of Q -exact and Q -closed terms:

$$g^2\mathcal{L} = \mathcal{L}_{\text{exact}} + \mathcal{L}_{\text{closed}} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 = Q\tilde{\mathcal{L}}_{\text{exact}} + \mathcal{L}_{\text{closed}}, \tag{3.8}$$

where g is coupling constant and $\tilde{\mathcal{L}}_{\text{exact}} = \tilde{\mathcal{L}}_{e,1} + \tilde{\mathcal{L}}_{e,2}$ is given by

$$\begin{aligned}
\tilde{\mathcal{L}}_{e,1} &= \text{Tr} \left(\lambda \left(\frac{1}{2}id + \frac{1}{2}[\bar{\mathcal{D}}_\mu, \mathcal{D}^\mu] + \frac{1}{24}[\bar{z}^{\mu\nu\rho\sigma}, z_{\mu\nu\rho\sigma}] \right) \right) \\
\tilde{\mathcal{L}}_{e,2} &= \text{Tr} \left(\frac{i}{4}\xi_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{1}{12\sqrt{2}}\xi^{\nu\rho\sigma}\mathcal{D}^\mu z_{\mu\nu\rho\sigma} \right)
\end{aligned}
\tag{3.9}$$

and $\mathcal{L}_{\text{closed}}$ is given by

$$\mathcal{L}_{\text{closed}} = \mathcal{L}_3 = \text{Tr} \frac{1}{2}\xi_{\mu\nu}\bar{\mathcal{D}}_\rho\xi^{\mu\nu\rho} + \frac{\sqrt{2}}{8}\xi_{\mu\nu}[\bar{z}^{\mu\nu\rho\sigma}, \xi_{\rho\sigma}] \tag{3.10}$$

By using the transformation properties of fields and the equation of motion auxiliary field d

$$(-id) = \frac{1}{2}[\bar{\mathcal{D}}_\mu, \mathcal{D}^\mu] + \frac{1}{24}[\bar{z}^{\mu\nu\rho\sigma}, z_{\mu\nu\rho\sigma}], \tag{3.11}$$

we obtain the Lagrangian expressed in terms of propagating degrees of freedom: ⁶

$$\begin{aligned}
\mathcal{L}_1 &= \text{Tr} \left(\frac{1}{2} \left(\frac{1}{2}[\bar{\mathcal{D}}_\mu, \mathcal{D}^\mu] + \frac{1}{24}[\bar{z}^{\mu\nu\rho\sigma}, z_{\mu\nu\rho\sigma}] \right)^2 + \lambda(\bar{\mathcal{D}}_\mu\psi^\mu + \frac{1}{24}[\bar{z}^{\mu\nu\rho\sigma}, \psi_{\mu\nu\rho\sigma}]) \right) \\
\mathcal{L}_2 &= \text{Tr} \left(\frac{1}{4}\bar{\mathcal{F}}_{\mu\nu}\mathcal{F}^{\mu\nu} + \xi_{\mu\nu}\mathcal{D}^\mu\psi^\nu + \frac{1}{12}|\mathcal{D}^\mu z_{\mu\nu\rho\sigma}|^2 + \frac{1}{12}\xi^{\nu\rho\sigma}\mathcal{D}^\mu\psi_{\mu\nu\rho\sigma} + \frac{1}{6\sqrt{2}}\xi^{\nu\rho\sigma}[\psi^\mu, z_{\mu\nu\rho\sigma}] \right) \\
\mathcal{L}_3 &= \text{Tr} \left(\frac{1}{2}\xi_{\mu\nu}\bar{\mathcal{D}}_\rho\xi^{\mu\nu\rho} + \frac{\sqrt{2}}{8}\xi_{\mu\nu}[\bar{z}^{\mu\nu\rho\sigma}, \xi_{\rho\sigma}] \right).
\end{aligned}
\tag{3.12}$$

The Q -invariance of the $\mathcal{L}_{\text{exact}}$ is obvious and follows from supersymmetry algebra $Q^2 = 0$. To show the invariance of Q -closed term requires the use of the Bianchi (or Jacobi identity for covariant derivatives) identity

$$\epsilon^{\sigma\mu\nu\rho}\bar{\mathcal{D}}_\mu\bar{\mathcal{F}}_{\nu\rho} = \epsilon^{\sigma\mu\nu\rho}[\bar{\mathcal{D}}_\mu, [\bar{\mathcal{D}}_\nu, \bar{\mathcal{D}}_\rho]] = 0 \tag{3.13}$$

and similar identity involving scalars. The action is expressed in terms of the twisted Lorentz multiplets, and the $\text{SO}(4)' \times \text{U}(1)$ symmetry is manifest. The Lagrangian eq. (3.12) emerges from the hypercubic and A_4^* lattice action at the tree level. This will be discussed after the following digression to superfield formulation of Marcus's twist.

⁶Notice that the splitting of the exact terms in Lagrangian into \mathcal{L}_1 and \mathcal{L}_2 is not identical to the one used by Marcus. The reason for the above splitting lies in the symmetries of the cut-off theory (A_4^* lattice theory) that will be discussed later.

3.1 The $\mathcal{Q} = 1$ (twisted) superfields formulation of $\mathcal{N} = 4$ SYM

In this section, we introduce a superfield notation for the the twisted $\mathcal{N} = 4$ SYM theory. The superfields are $\text{SO}(4)'$ multiplets. This is so since the manifest supersymmetry is a scalar and exactly nilpotent. Consequently, different components of a multiplet (unlike the usual supersymmetry multiplets in four dimensions) reside in the same representation of twisted rotation group.

The supermultiplets are all in integer spin representations of $\text{SO}(4)'$. The superfields are a scalar fermi multiplet $\Lambda(x)$ transforming as $(1)_{\frac{1}{2}}$, a vector multiplet $\mathbf{Z}^\mu(x)$ transforming as $(4)_0$, a two-form fermi multiplet $\Xi_{\mu\nu}(x)$ transforming as $(6)_{\frac{1}{2}}$, a three-form fermi multiplet $\Xi^{\mu\nu\rho}(x)$ in $(4)_{-\frac{1}{2}}$, and a four form $\mathbf{Z}_{\mu\nu\rho\sigma}(x)$ in $(1)_1$ representation. There are also two types of supersymmetry singlets, a vector $\bar{z}_\mu(x)$ in $(4)_0$ and a four form $\bar{z}^{\mu\nu\rho\sigma}(x)$ transforming as $(1)_{-1}$. The scalar $\mathcal{Q} = 1$ off-shell supersymmetry transformations can then be realized in terms of these superfields as

$$\begin{aligned}
 \Lambda(x) &= \lambda(x) - \theta id(x), \\
 \mathbf{Z}^\mu(x) &= z^\mu(x) + \sqrt{2}\theta\psi^\mu(x), & \bar{z}_\mu(x), \\
 \Xi_{\mu\nu}(x) &= \xi_{\mu\nu}(x) - i\theta\bar{\mathcal{F}}_{\mu\nu}(x), \\
 \Xi^{\nu\rho\sigma}(x) &= \xi^{\nu\rho\sigma}(x) + \sqrt{2}\theta\bar{\mathcal{D}}_\mu\bar{z}^{\mu\nu\rho\sigma}(x), \\
 \mathbf{Z}_{\mu\nu\rho\sigma}(x) &= z_{\mu\nu\rho\sigma}(x) + \sqrt{2}\theta\psi_{\mu\nu\rho\sigma}(x), & \bar{z}^{\mu\nu\rho\sigma}(x).
 \end{aligned} \tag{3.14}$$

These superfields should be useful in formulating the $\mathcal{N} = 4$ SYM not only on \mathbb{R}^4 , but on arbitrary curved four-manifold M^4 , mainly because they are based on scalar supersymmetry. By introducing the super-covariant derivative;

$$\mathcal{D}^\mu = \partial^\mu + \sqrt{2}\mathbf{Z}^\mu = \mathcal{D}^\mu + 2\theta\psi^\mu \tag{3.15}$$

we can also define the field strength multiplet

$$\mathcal{F}^{\mu\nu} = -i[\mathcal{D}^\mu, \mathcal{D}^\nu] = \mathcal{F}^{\mu\nu} - 2i\theta(\mathcal{D}^\mu\psi^\nu - \mathcal{D}^\nu\psi^\mu) \tag{3.16}$$

transforming as $(6)_0$ under $\text{SO}(4)' \times \text{U}(1)$. In terms of the $\mathcal{Q} = 1$ superfields, the action of the $\mathcal{N} = 4$ SYM theory on \mathbb{R}^4 can be expressed as

$$\begin{aligned}
 S = \frac{1}{g^2} \text{Tr} \int d^4x d\theta \left(& -\frac{1}{2}\Lambda\partial_\theta\Lambda - \Lambda\left(\frac{1}{2}[\bar{\mathcal{D}}_\mu, \mathcal{D}^\mu] + \frac{1}{24}[\bar{z}^{\mu\nu\rho\sigma}, \mathbf{Z}_{\mu\nu\rho\sigma}]\right) \right. \\
 & + \frac{i}{4}\Xi_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{1}{12\sqrt{2}}\Xi^{\nu\rho\sigma}\mathcal{D}^\mu\mathbf{Z}_{\mu\nu\rho\sigma} \left. \right) \\
 & + \frac{1}{2}\Xi_{\mu\nu}\bar{\mathcal{D}}_\rho\Xi^{\mu\nu\rho} + \frac{\sqrt{2}}{8}\Xi_{\mu\nu}[\bar{z}^{\mu\nu\rho\sigma}, \Xi_{\rho\sigma}]
 \end{aligned} \tag{3.17}$$

The last line is not integrated over the superspace and is the Q -closed term discussed above. Its θ component vanishes because of Jacobi identities, hence it is supersymmetric. Notice that the three lines of this action respectively corresponds to $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ in eq. (3.12).

3.2 Hypercubic lattice

The action eq. (3.12), or equivalently eq. (3.17), expressed in terms of integer spin representations (p -forms) of $SO(4)'$ arises naturally from orbifold lattices [8, 36]. Recall that the fundamental cell of the hypercubic lattice contains one site, four links, six faces, four cubes and one hypercube, collectively named as p -cells. A p -form tensor fermion is associated with a p -cell on the hypercubic lattice. The complex vectors of $SO(4)'$ are associated with the link fields. Finally, the two scalars (four-forms) are associated with the four-cell as can be deduced from the supersymmetry algebra.

The action eq. (3.17) with manifest scalar supersymmetry (in fact, possessing all sixteen supersymmetries) admits a discretization to a hypercubic lattice in which one preserves the scalar supercharge. The hypercubic lattice action is given in [36]. The rules of latticization are natural and given by Catterall (except the rule which requires complexification of the fields. Our bosons and fermions are already complex and oriented.) [16, 35]. For our purpose, it suffices to understand the transformations given in eq. (3.6). The local transformations in eq. (3.6) remain the same, modulo the trivial substitution of space-time position x with a discrete lattice position index \mathbf{n} . There are two types of semi-local transformation. The first one is $Q\xi_{\mu\nu}(x) = -i\overline{\mathcal{F}}_{\mu\nu}(x)$. This translates to $Q\xi_{\mu\nu,\mathbf{n}} = -2(\overline{z}_{\mu,\mathbf{n}+\mathbf{e}_\nu}\overline{z}_{\nu,\mathbf{n}} - \overline{z}_{\nu,\mathbf{n}+\mathbf{e}_\mu}\overline{z}_{\mu,\mathbf{n}})$. The right hand side is the square root of the usual Wilson plaquette term. Similarly, $\mathcal{F}^{\mu\nu}(x)$ becomes $z_{\mathbf{n}}^\mu z_{\mathbf{n}+\mathbf{e}_\mu}^\nu - z_{\mathbf{n}}^\nu z_{\mathbf{n}+\mathbf{e}_\nu}^\mu$.⁷ The second transformation $Q\xi^{\nu\rho\sigma}(x) = \sqrt{2}\overline{\mathcal{D}}_\mu\overline{z}^{\mu\nu\rho\sigma}(x)$ translates into $Q\xi_{\mathbf{n}}^{\nu\rho\sigma} = 2(\overline{z}_{\mu,\mathbf{n}-\mathbf{e}_{\mu\nu\rho\sigma}}\overline{z}_{\mathbf{n}}^{\mu\nu\rho\sigma} - \overline{z}_{\mathbf{n}+\mathbf{e}_\mu}^{\mu\nu\rho\sigma}\overline{z}_{\mu,\mathbf{n}})$ where $\mathbf{e}_{\mu\nu\rho\sigma} = \sum_\zeta \mathbf{e}_\zeta$ and $(\mathbf{e}_\mu)_\nu = \delta_{\mu\nu}$ are the cartesian unit vectors. It is appropriate to parametrize the complex link fields z^μ as

$$z_{\mathbf{n}}^\mu = \frac{1}{\sqrt{2}a} e^{a(S_{\mu,\mathbf{n}} + iV_{\mu,\mathbf{n}})}, \tag{3.18}$$

where a is the lattice spacing. Substituting these into, for example, $\overline{\mathcal{F}}_{\mu\nu}\mathcal{F}^{\mu\nu}$ produce a complexified Wilson action. This parametrization differs from the ones used in [37, 8]. However, the difference in the continuum is in the irrelevant operators, suppressed by powers of the lattice spacing. This prescription generates the lattice actions discussed in detail in [36, 8] and we will not duplicate it here. Instead, we want to comment on the emergence of large global chiral symmetries, the R -symmetry, in the continuum of orbifold lattices.

In lattice QCD, Poincaré invariance emerges in the continuum without any fine tuning, due to the point group symmetry, discrete translation symmetry and gauge invariance. The Poincaré violating relevant and marginal operators are usually forbidden due to these symmetries, and we recover the Poincaré invariant target theory. In our case, the continuum limit of the hypercubic lattice at tree level, by construction, reproduces the target theory with a twisted Lorentz invariance $SO(4)'$. The $U(1)_R$ symmetry is exact on the hypercubic lattice, and hence it is exact in the continuum. The $SO(4)' \times U(1)$ invariant target theory is a redefinition of the physical $\mathcal{N} = 4$ SYM theory, which possesses a Lorentz symmetry group

⁷Recall that the usual Wilson action may also be written as $S = \sum_{\mathbf{n}} \text{Tr} |U_{\mu,\mathbf{n}}U_{\nu,\mathbf{n}+\mathbf{e}_\mu} - U_{\nu,\mathbf{n}}U_{\mu,\mathbf{n}+\mathbf{e}_\nu}|^2$ where the quantity in modulus is the field strength and is indeed the square root of a plaquette.

and a large R -symmetry, $SO(4) \times SO(6)$. The twisting obscures the large R -symmetry. However, knowing how $SO(4)'$ arises, we see that there is really an $SO(4)_E \times SO(4)_R$ behind what appears to be a twisted Lorentz symmetry. This means the large R -symmetry group arises from the lattice hand in hand with Lorentz symmetry. This happens to be so since the point symmetry group of the lattice is a subgroup of the diagonal subgroup of $SO(4)_E \times SO(4)_R$. Of course, the full R -symmetry is $SO(6)$ and the above tree level argument only explains $SO(4)_R \times U(1)$ subgroup of it. We will, nevertheless, be content with it.

3.3 What does the A_4^* lattice knows about twisting?

In this section, we want to explain an elegant relation between A_4^* lattice and the twisted continuum theory eq. (3.12).⁸ The A_4^* lattice for $\mathcal{N} = 4$ SYM theory is introduced in ref. [8] and arises as the most symmetrical lattice arrangement in the moduli space of orbifold lattice theory. In particular, it is more symmetric than the hypercubic lattice. Higher symmetry is an important virtue when the renormalization and the quantum continuum limit of the lattice theory is addressed. When considering the radiative corrections, the relevant and marginal operators will be restricted by the symmetries of the underlying theory. Therefore, fewer relevant and marginal operators will exist for the more symmetric spacetime lattice. For lower dimensional examples, the combination of lattice point group symmetry, the exact supersymmetry and superrenormalizability are used to show that the desired target theories are attained with no or few fine tunings at the quantum level. We hope that the techniques of this section can eventually be used in addressing the important problem of renormalization of $\mathcal{N} = 4$ in $d = 4$ dimensions. Our aim here is different, and in fact more modest - namely showing the relation between the A_4^* lattice and Marcus's twist. Our analysis is at the tree level. We show this relation by finding the irreducible representations of the point group symmetry of the lattice action, and by identifying them with the ones of the twisted Lorentz group $SO(4)'$.

We have already seen the field distribution on the hypercubic lattice and identified lattice p -cell fields with p -form tensors in the continuum. The situation for A_4^* is a little more subtle and requires basic representation and character theory for finite groups. The generalization to other dimensions for which target theory is $\mathcal{Q} = 16$ and lattice is A_d^* is obvious.

The A_4^* lattice is generated by the fundamental weights, or equivalently by the weights of defining representation $SU(5)$. A specific basis for A_4^* lattice is given in the form of five, four dimensional lattice vectors:

$$\begin{aligned} \mathbf{e}_1 &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{20}} \right) \\ \mathbf{e}_2 &= \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{20}} \right) \\ \mathbf{e}_3 &= \left(0, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{20}} \right) \end{aligned}$$

⁸This techniques used in this section borrows from unpublished notes of David B. Kaplan on A_3^* lattices for spatial lattice construction of $\mathcal{N} = 4$ SYM in the context of renormalization. I would like to thank him for sharing them with me.

$$\begin{aligned} \mathbf{e}_4 &= (0, 0, -\frac{3}{\sqrt{12}}, \frac{1}{\sqrt{20}}) \\ \mathbf{e}_5 &= (0, 0, 0, -\frac{4}{\sqrt{20}}). \end{aligned} \tag{3.19}$$

These vectors satisfy the relations

$$\sum_{m=1}^5 \mathbf{e}_m = 0, \quad \mathbf{e}_m \cdot \mathbf{e}_n = \left(\delta_{mn} - \frac{1}{5} \right), \quad \sum_{m=1}^5 (\mathbf{e}_m)_\mu (\mathbf{e}_m)_\nu = \delta_{\mu\nu}. \tag{3.20}$$

The lattice vectors eq. (3.19) connect the center of a 4-simplex to its five corners and are simply related to the SU(5) weights of the $\mathbf{5}$ representation. The unit cell of the lattice is a compound of two 4-simplex as in the $\mathbf{5}$ and $\bar{\mathbf{5}}$ representations of SU(5).⁹

The matter content of A_4^* lattice theory is most easily described in terms of representations of SU(5). The ten bosonic degrees of freedom are labeled as $z^m \oplus \bar{z}_m = \mathbf{5} \oplus \bar{\mathbf{5}}$, and the sixteen fermions are presented as $\lambda \oplus \psi^m \oplus \xi_{mn} = \mathbf{1} \oplus \mathbf{5} \oplus \bar{\mathbf{10}}$. The z^m, ψ^m fields reside on the links connecting the center of a 4-simplex to its five corners, which are labeled by \mathbf{e}_m . The \bar{z}_m reside on the links along $-\mathbf{e}_m$. The ten fermions ξ_{mn} , and ten composite antiholomorphic bosonic fields $\bar{E}_{mn} = [\bar{z}_m, \bar{z}_n]$ reside on $-\mathbf{e}_m - \mathbf{e}_n$ directed toward the ten sides of the 4-simplex and finally the singlet λ resides on the site. For more details on the A_4^* lattice, see ref. [8]. The point group symmetry of the action is permutation group S_5 , the Weyl group of SU(5). Notice that inversion is not a symmetry, since there is no $\bar{\mathbf{5}}$ representation in fermionic sector.

Let us reexpress the A_4^* lattice action for $\mathcal{N} = 4$ theory as a sum of Q -exact \mathcal{L}_1 and \mathcal{L}_2 , and Q -closed \mathcal{L}_3 terms. Because of symmetry reasons and to ease the comparison with eq. (3.12), we present it as $g^2 \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$ where

$$\begin{aligned} \mathcal{L}_1 &= \sum_{\mathbf{n}} Q \text{Tr} \lambda_{\mathbf{n}} \left(\frac{1}{2} id_{\mathbf{n}} + (\bar{z}_{m, \mathbf{n} - \mathbf{e}_m} z_{\mathbf{n} - \mathbf{e}_m}^m - z_{\mathbf{n}}^m \bar{z}_{m, \mathbf{n}}) \right) \\ \mathcal{L}_2 &= \sum_{\mathbf{n}} Q \text{Tr} \frac{1}{2} \xi_{mn, \mathbf{n}} (z_{\mathbf{n}}^m z_{\mathbf{n} + \mathbf{e}_m}^n - z_{\mathbf{n}}^n z_{\mathbf{n} + \mathbf{e}_n}^m) \\ \mathcal{L}_3 &= \sum_{\mathbf{n}} \frac{\sqrt{2}}{8} \epsilon^{mnpqr} \text{Tr} \xi_{mn, \mathbf{n}} (\bar{z}_{p, \mathbf{n} - \mathbf{e}_p} \xi_{qr, \mathbf{n} + \mathbf{e}_m + \mathbf{e}_n} - \xi_{qr, \mathbf{n} - \mathbf{e}_q - \mathbf{e}_r} \bar{z}_{p, \mathbf{n} + \mathbf{e}_m + \mathbf{e}_n}). \end{aligned} \tag{3.21}$$

The supersymmetry transformations of the lattice fields are given by

$$\begin{aligned} Q \lambda_{\mathbf{n}} &= -id_{\mathbf{n}}, & Q d_{\mathbf{n}} &= 0 \\ Q z_{\mathbf{n}}^m &= \sqrt{2} \psi_{\mathbf{n}}^m, & Q \psi_{\mathbf{n}}^m &= 0 \\ Q \bar{z}_{m, \mathbf{n}} &= 0 \end{aligned}$$

⁹Three dimensional counterpart is A_3^* lattice, the body centered cubic lattice. The unit cell should be regarded as a compound $\mathbf{4}$ and $\bar{\mathbf{4}}$ representations of SU(4). The $\mathbf{4}$ ($\bar{\mathbf{4}}$) is generated by four, three dimensional vectors, \mathbf{e}_m ($-\mathbf{e}_m$) with $m = 1, \dots, 4$, which can be obtained by removing the fourth component from eq. (3.19), i.e. by dimensional reduction. The compound of two tetrahedron, (the $\mathbf{4}$ and $\bar{\mathbf{4}}$) is the famous *Stella Octangula* of Kepler, the simplest polyhedral compound in three dimensions. The cube arises as the convex hull of this object. Two dimensional counterpart is A_2^* lattice, triangular lattice. It can be thought as $\mathbf{3}$ and $\bar{\mathbf{3}}$ representation of SU(3). The $\mathbf{3}$ ($\bar{\mathbf{3}}$) is generated by three, two dimensional vectors, \mathbf{e}_m ($-\mathbf{e}_m$) with $m = 1, \dots, 3$, which can be obtained by removing the third and the fourth component from eq. (3.19). The A_4^* lattice can be visualized similarly.

classes:	(1)	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
sizes:	1	10	20	30	24	15	20
$\tilde{\chi}_1$	1	1	1	1	1	1	1
$\tilde{\chi}_2$	1	-1	1	-1	1	1	-1
$\tilde{\chi}_3$	4	2	1	0	-1	0	-1
$\tilde{\chi}_4$	4	-2	1	0	-1	0	1
$\tilde{\chi}_5$	5	-1	-1	1	0	1	-1
$\tilde{\chi}_6$	5	1	-1	-1	0	1	1
$\tilde{\chi}_7$	6	0	0	0	1	-2	0

Table 2: The character table of S_5 , the point symmetry group of A_4^* lattice. The even permutations are spacetime rotations, the odd permutations involves parity operations and hence improper rotations.

$$Q\xi_{mn,\mathbf{n}} = -2(\bar{z}_{m,\mathbf{n}+\mathbf{e}_n}\bar{z}_{n,\mathbf{n}} - \bar{z}_{n,\mathbf{n}+\mathbf{e}_m}\bar{z}_{m,\mathbf{n}}). \tag{3.22}$$

Clearly, the S_5 singlet supersymmetry Q is nilpotent, $Q^2 = 0$. In the rest of this section, we show the transmutation of action eq. (3.21) into eq. (3.12) by using the representation theory of S_5 . To reduce the clutter, we suppress the lattice indices which transform in an obvious way under the point group symmetry.

The physical point group symmetry of the lattice is isomorphic to permutation group S_5 . The character table and conjugacy classes of S_5 are given in table 2. The group has $5! = 120$ elements and seven conjugacy classes shown in table 2. The symmetry of the lattice action is composed of the elements of S_5 . It is easy to show that even permutations with determinant one (the $\tilde{\chi}_2$ representation) are pure rotational symmetries of the action. We see from table 2 that the odd permutations has determinant minus one (the $\tilde{\chi}_2$ representation), and are not proper elements of $SO(4)'$. Hence, we consider A_5 , the rotation subgroup of S_5 , also called alternating group of degree five. The A_5 is the discrete subgroup of proper rotations $SO(4)'$

$$A_5 = S_5/Z_2 \in SO(4)'$$

and we classify fields under A_5 . Also, as noted in [8], the odd permutations are symmetries if accompanied with a fermion phase redefinitions $\xi \rightarrow i\xi$, $\psi \rightarrow -i\psi$, and $\lambda \rightarrow i\lambda$. The odd permutations do not commute with supersymmetry as the field redefinition treats the components of a supermultiplet differently.

To classify fields under A_5 , we consider the group action from each of the five conjugacy classes. The character table of A_5 can be deduced from S_5 ¹⁰ and is given in table 3. By

¹⁰The conjugacy classes of A_5 can easily be read off from the character table and conjugacy classes of S_5 . The character table of S_5 is given in table 2. The conjugacy classes of S_5 are the physical symmetries of the 4-simplex. The conjugacy classes (which is formed of only even permutations) and the sizes of representations of A_5 are given in table 3. Notice that in A_5 , the 5-cycles splits into two types. (12345) and (21345). It is easy to see that only the odd permutations (which are absent in A_5 , but present in S_5)

classes:	(1)	(123)	(12345)	(21345)	(12)(34)
sizes:	1	20	12	12	15
χ_1	1	1	1	1	1
χ_2	4	1	-1	-1	0
χ_3	5	-1	0	0	1
χ_4	3	0	$\frac{(1+\sqrt{5})}{2}$	$\frac{(1-\sqrt{5})}{2}$	-1
χ_5	3	0	$\frac{(1-\sqrt{5})}{2}$	$\frac{(1+\sqrt{5})}{2}$	-1

Table 3: The character table of A_5 , the rotation subgroup of S_5 . The pure rotational symmetries of A_4^* lattice.

Operation	z^1, z^2, z^3, z^4, z^5	$\chi(g_{(rep)})$
(1)	z^1, z^2, z^3, z^4, z^5	5
(123)	z^3, z^1, z^2, z^4, z^5	2
(12345)	z^5, z^1, z^2, z^3, z^4	0
(13452)	z^2, z^5, z^1, z^3, z^4	0
(12)(34)	z^2, z^1, z^4, z^3, z^5	1

Table 4: A representative of each conjugacy class and their action on the site and link fields are shown in the table. The five link fermions ψ^m transform in the same way with z^m . The transformation of ten fermions ξ_{mn} can be deduced from the antisymmetric product representation of \bar{z}_m with itself.

choosing a representative from each conjugacy class, we calculate the character of the corresponding group element. In table 4, we show how an element from each class acts on the link field and calculate the character $\chi(g) = \text{Tr}(O(g))$, where g is a representative of each class and O is a matrix representation of the operation. Since the character is a class function, it is independent of representative. A simple calculation for all the lattice fields yields

$$\begin{aligned}
 \chi(\lambda) &= \chi(d) = (1, 1, 1, 1, 1) \sim \chi_1 \\
 \chi(z^m) &= \chi(\psi^m) = \chi(\bar{z}_m) = (5, 2, 0, 0, 1) \sim \chi_2 \oplus \chi_1 \\
 \chi(\xi_{mn}) &= \chi(\bar{E}_{mn}) = (10, 1, 0, 0, -1) \sim \chi_2 \oplus \chi_4 \oplus \chi_5.
 \end{aligned}
 \tag{3.23}$$

By inspecting the character table, we observe that the link fields are indeed in reducible rep-

can take an element of one conjugacy class to the other. Hence there are two distinct conjugacy classes for five-cycles in A_5 .

The characters for A_5 can also be deduced from the ones of S_5 . Since the odd permutations are absent in A_5 , the sign representation of S_5 reduce to trivial representation $\tilde{\chi}_2|_{A_5} = \chi_1$. Also, noticing the relations $\tilde{\chi}_3\tilde{\chi}_2 = \tilde{\chi}_4$, $\tilde{\chi}_5\tilde{\chi}_2 = \tilde{\chi}_6$, $\tilde{\chi}_7\tilde{\chi}_2 = \tilde{\chi}_7$, we see that $\tilde{\chi}_3|_{A_5} = \tilde{\chi}_4|_{A_5} = \chi_2$, $\tilde{\chi}_5|_{A_5} = \tilde{\chi}_6|_{A_5} = \chi_3$. Finally, the $\tilde{\chi}_7$ is reducible in A_5 . From the relation, $\tilde{\chi}_7\tilde{\chi}_2 = \tilde{\chi}_7$, we see that $\tilde{\chi}_7$ is zero for all odd permutations. It splits as $\tilde{\chi}_7|_{A_5} = \chi_4 + \chi_5$ into two three dimensional representations. As a physical consequence, unlike S_5 , the A_5 can not distinguish a scalar from pseudo-scalar and a vector from a pseudo-vector.

representations of the point group symmetry A_5 . The site multiplet is in trivial representation.

$$\chi(\lambda) = \chi(d) = \chi_1. \tag{3.24}$$

The five link fields splits as

$$\chi(z^m) = \chi(\psi^m) = \chi(\bar{z}_m) = \chi_2 \oplus \chi_1, \tag{3.25}$$

into a singlet representation and a four dimensional representation. Similarly, the ten fermions ξ_{mn} decompose into a four and two three dimensional representations as

$$\chi(\xi_{mn}) = \chi(\bar{E}_{mn}) = \chi_2 \oplus \chi_4 \oplus \chi_5 \tag{3.26}$$

One can also show that the product of fields which make the $E^{mn} = [z^m, z^n]$ function transforms as

$$\chi(E^{mn}) = [\chi(z^m) \otimes \chi(z^n)]_{A.S.} = \chi_2 \oplus \chi_4 \oplus \chi_5 \tag{3.27}$$

Notice that the splitting of $\chi(\xi_{mn})$ and $\chi(E^{mn})$ in S_5 is $\tilde{\chi}_3 \oplus \tilde{\chi}_7$ into a four and six dimensional representation. Under A_5 , $\tilde{\chi}_7|_{A_5} = \chi_4 \oplus \chi_5$ splits further because of lower symmetry. We observe that the elementary fermionic and bosonic degrees of freedom split into irreducible representations as

$$\begin{aligned} \text{fermions} &\rightarrow 2\chi_1 \oplus 2\chi_2 \oplus [\chi_4 \oplus \chi_5] = 2(1) \oplus 2(4) \oplus [3 \oplus 3'] \\ \text{bosons} &\rightarrow 2\chi_1 \oplus 2\chi_2 = 2(1) \oplus 2(4), \end{aligned} \tag{3.28}$$

where the dimension of the corresponding irreducible representations is written explicitly. This is indeed the branching of fermions and bosons under the twisted Lorentz symmetry $SO(4)'$ discussed in section 3. It is easy to identify the continuum fields (which transform under the irreducible representations of twisted rotation symmetry $SO(4)'$) with the irreducible representation of the discrete rotations on the lattice. The two scalar fermions of the continuum theory are associated with the two singlet (χ_1) fermions on the lattice. Similarly, the vector and three form of the continuum are the two four dimensional χ_2 representation. Finally, the six fermions (in two index antisymmetric representation) of the continuum theory 6 of $SO(4)'$ reside in the two three dimensional representation $\chi_4 \oplus \chi_5$ of the A_5 , or better in $\tilde{\chi}_7$ of S_5 . The self-dual and antiselfdual splitting of 6 into $[(3, 1) \oplus (1, 3)]$ representations takes place in the spin group of $SO(4)'$ and these two three dimensional representation is not related to $\chi_4 \oplus \chi_5$ of A_5 . The bosonic degrees of freedom work similarly.

How can we compute these irreducible representations explicitly? For example, for link fields z^m , what does the splitting $\chi_2 \oplus \chi_1$ mean? Recall that under a group operation (see table 4), $z^m \rightarrow O^{mn}(g)z^n$. Dropping all the indices, $z' = Oz$. The fact that the group action on the link field is reducible means there is a similarity transformation which takes all of the $O(g)$ into a block diagonal form. In this case, two blocks have sizes 1×1 and 4×4 . This naturally splits the z_m vector space into two components of size one and four, which never mixes under group action. It is easy to guess the singlet representation: it is $\frac{1}{\sqrt{5}} \sum_{m=1}^5 z^m$. Now, let us introduce a 5×5 orthogonal matrix \mathcal{E} that block-diagonalizes

$O(g)$ for all $g \in A_5$. Then we have $(\mathcal{E}^{-1}z') = (\mathcal{E}^{-1}O\mathcal{E})(\mathcal{E}^{-1}z)$. A little bit work shows that the \mathcal{E} matrix can be expressed in terms of components of the basis vectors \mathbf{e}_m ¹¹

$$\mathcal{E}_{m\mu} = (\mathbf{e}_m)_\mu, \quad \mathcal{E}_{m5} = \frac{1}{\sqrt{5}}. \quad (3.29)$$

The matrix \mathcal{E}_{mn} forms a bridge between the irreducible representation of A_5 and the representations of the twisted Lorentz group $SO(4)'$. Thus, we obtain the following relations dictated by symmetry arguments:

$$\mathcal{E}_{m\mu} z_{\mathbf{n}}^m = z^\mu(x) \quad \mathcal{E}_{m5} z_{\mathbf{n}}^m = \epsilon_{\mu\nu\rho\sigma} z^{\mu\nu\rho\sigma}(x)/24 \quad (3.30)$$

For fermions ξ_{mn} , it is easy to show that the continuum fields are $\xi_{mn,\mathbf{n}} \mathcal{E}_{m\mu} \mathcal{E}_{n\nu} = \xi_{\mu\nu}(x)$ and $\xi_{mn,\mathbf{n}} \mathcal{E}_{m\mu} \mathcal{E}_{n5} = \epsilon_{\mu\nu\rho\sigma} \xi^{\nu\rho\sigma}(x)/6$. Similarly, the antiholomorphic function $\bar{E}_{mn,\mathbf{n}}$ splits into $\bar{\mathcal{F}}_{\mu\nu}(x)$ and $\bar{\mathcal{D}}_\mu \bar{z}^{\mu\nu\rho\sigma}(x)$. This completes our discussion of the relation between the A_4^* lattice action and twisted theory eq. (3.12). The continuum limit of the Lagrangian eq. (3.21) at tree level reproduce the twisted theory eq. (3.12).

3.4 Connection with Catterall's formulation

Another recent proposal for lattice regularization of $\mathcal{N} = 4$ SYM theory had been introduced by S. Catterall. In this section, we want to briefly mention the relation between the two approaches. In fact, upon realizing the fact that the orbifold lattice produces Marcus's twist, this is an obvious task. Catterall already provides the mapping between the Lagrangian eq. (3.12) and the action he employs in latticization [17]. Here, we construct the relation only in the sense of supersymmetry subalgebras that are manifest in these two lattice constructions.

Let us rename the twist introduced in the previous section as an A-type twist. In fact, there is another scalar supersymmetry, associated with Poincaré dual of the 4-form Grassmann $*\psi^{(4)}$. We could have chosen $\bar{Q} = *Q^{(4)} = \frac{1}{4!} \epsilon_{\mu\nu\rho\sigma} Q^{\mu\nu\rho\sigma}$ as the manifest scalar supersymmetry. We call this B-type supersymmetry. To make the comparison with the Catterall lattice formulation, it is convenient to dualize the 3-form and 4-forms fields to vectors and scalars respectively.

$$\frac{1}{4!} \epsilon^{\mu\nu\rho\sigma} (z_{\mu\nu\rho\sigma}, \psi_{\mu\nu\rho\sigma}) = (z, \psi) \quad \frac{1}{4!} \epsilon_{\mu\nu\rho\sigma} \bar{z}^{\mu\nu\rho\sigma} = \bar{z}, \quad \frac{1}{3!} \epsilon_{\mu\nu\rho\sigma} \xi^{\nu\rho\sigma} = \chi_\mu, \quad (3.31)$$

The continuum on-shell A-type supersymmetry transformation are given by¹²

$$Q\lambda = -([\bar{z}, z] + \frac{1}{2}[\bar{\mathcal{D}}_\mu, \mathcal{D}^\mu])$$

¹¹This relation is true for all A_d^* lattices. In arbitrary d , we have $d+1$ linearly dependent d -dimensional vectors satisfying $\mathbf{e}_m \cdot \mathbf{e}_n = \delta_{mn} - \frac{1}{d+1}$ and $\sum_{m=1}^{d+1} \mathbf{e}_m = 0$. The block diagonalization matrix \mathcal{E} takes the form $\mathcal{E}_{m\mu} = (\mathbf{e}_m)_\mu$, $\mathcal{E}_{m,d+1} = \frac{1}{\sqrt{d+1}}$. The matrix \mathcal{E} connects the irreducible representations of S_{d+1} to the ones of $SO(d)'$.

¹²The transformations in eq. (3.32), eq. (3.33) and eq. (3.34) can easily be read off of the transformation given in eq. (4.9) of ref. [8] by using the substitution $z^\mu \rightarrow \mathcal{D}^\mu/\sqrt{2}$ and $\bar{z}_\mu \rightarrow \bar{\mathcal{D}}_\mu/\sqrt{2}$.

$$\begin{aligned}
Qz^\mu &= \sqrt{2} \psi^\mu, & Q\psi^\mu &= 0 \\
Q\bar{z}_\mu &= 0 \\
Q\xi_{\mu\nu} &= -i\bar{\mathcal{F}}_{\mu\nu} \\
Q\chi_\mu &= \sqrt{2}\bar{\mathcal{D}}_\mu\bar{z} \\
Qz &= \sqrt{2}\psi, & Q\psi &= 0 \\
Q\bar{z} &= 0
\end{aligned} \tag{3.32}$$

and similarly the on-shell B-type transformations are

$$\begin{aligned}
\bar{Q}\lambda &= 0 \\
\bar{Q}z^\mu &= 0, & \bar{Q}\psi^\mu &= \sqrt{2}\mathcal{D}^\mu\bar{\phi} \\
\bar{Q}\bar{z}_\mu &= -\sqrt{2}\chi_\mu, \\
\bar{Q}\xi_{\mu\nu} &= -i(\frac{1}{2}\epsilon_{\mu\nu\rho\sigma})\mathcal{F}^{\rho\sigma} \\
\bar{Q}\chi_\mu &= 0 \\
\bar{Q}z &= \sqrt{2}\lambda, & \bar{Q}\psi &= (\frac{1}{2}[\bar{\mathcal{D}}_\mu, \mathcal{D}^\mu] - [\bar{z}, z]) \\
\bar{Q}\bar{z} &= 0
\end{aligned} \tag{3.33}$$

Notice that both Q and \bar{Q} are nilpotent: $Q^2 = 0$, $\bar{Q}^2 = 0$, up to the use of equation of motion. As we have seen in the previous section, an off-shell completion is possible by introducing an auxiliary field d . A linear combination of A and B-type scalar supersymmetries is the exact manifest supersymmetry that is utilized in Catterall's formulation. Since the U(1) charges of these two supercharges are equal, we can add them without upsetting this symmetry. Using the supercharge

$$\tilde{Q} = \frac{Q^{(0)} + *Q^{(4)}}{\sqrt{2}} = \frac{Q + \bar{Q}}{\sqrt{2}} \tag{3.34}$$

we observe that the off-shell \tilde{Q} -action on fields are given by

$$\begin{aligned}
\tilde{Q}z^\mu &= \psi^\mu, & \tilde{Q}\psi^\mu &= \mathcal{D}^\mu\bar{z}, \\
\tilde{Q}\bar{z}_\mu &= -\chi_\mu, & \tilde{Q}\chi_\mu &= -\bar{\mathcal{D}}_\mu\bar{z} \\
\tilde{Q}z &= (\psi + \lambda), & \tilde{Q}(\psi + \lambda) &= -\sqrt{2}[\bar{z}, z] \\
\tilde{Q}\bar{z} &= 0, & \tilde{Q}(\psi - \lambda) &= \frac{1}{\sqrt{2}}[\bar{\mathcal{D}}_\mu, \mathcal{D}^\mu] \\
\tilde{Q}\xi_{\mu\nu} &= \frac{-i}{\sqrt{2}}(\bar{\mathcal{F}}_{\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\rho\sigma})
\end{aligned} \tag{3.35}$$

The \tilde{Q} transformation satisfies

$$\tilde{Q}^2 \cdot = \delta_{\bar{z}} \cdot, \tag{3.36}$$

which can be seen by using equations of motion. Here, $\delta_{\bar{z}}$ is a field dependent infinitesimal gauge transformation. Notice that \tilde{Q} is not exactly nilpotent, but nilpotent up to a gauge rotation. Catterall employs eq. (3.34) as the exact manifest supersymmetry on the lattice. Naturally, continuum actions in terms of propagating degrees of freedom can be easily mapped into each other. However, the number of bosonic off-shell degrees of freedom are not same in the two formulations. This can be understood by working the off-shell

completion of the supersymmetry algebra eq. (3.34). It is different from eq. (3.6) and necessitates introducing a two form auxiliary field. For the details of this construction, see [17]. I do not know the precise relation with the formulation of Sugino [21], but similar considerations may hold. However, I want to comment on the merit of having more than one formulation in a somewhat idiosyncratic way, by using reasonings from the calculations of topological correlators in the continuum formulation.

3.5 The fermion sign problem and topological correlators

The extended supersymmetric gauge theories shown in table 1 in general have a fermion sign problem even in continuum. In the case of $\mathcal{N} = 4$ SYM theory, the source of the sign problem can be traced to the Yukawa interactions, and therefore to nonvanishing field configurations of scalars. Conversely, in $\mathcal{N} = 1$ SYM in $d = 4$ dimensions, a theory without scalars, the positivity of the fermion Pfaffian can be proven. Here, I will argue that for a very restricted class of observables, the fermion sign problem should not be a problem. Similar considerations may hold in some lattice formulations as well. Unfortunately, this class is really small and the consideration of this section does not mean much for the full set of correlators of the physical theory. However, one can also pursue a more optimistic complementary logic [29]. Since many things are known or conjectured about the $\mathcal{N} = 4$ or other highly supersymmetric target theories, this data can be used to make progress in the understanding of the sign problem. After all, the sign problem arise because of inadequacy of the path integral, and is not a pathology of the theory. The reason that one can evade sign problem for topological correlators is a localization property of the path integral that we explain below. The ideas in this section borrows directly from the Witten's classic construction of topological field theory [1] and adopts the arguments there to the $\mathcal{N} = 4$ SYM theory.

The transformations eq. (3.35) look rather similar to the ones introduced by Witten in the study of the Donaldson theory [1]. Indeed, the supersymmetry algebras are identical, $\tilde{Q}^2 = \delta_{\bar{z}}$. The difference is in the field content. Witten considers the twist of $\mathcal{N} = 2$ theory, an asymptotically free theory in $d = 4$ dimensions, and addresses questions about the topological correlators (in the sense of \tilde{Q}) in the twisted theory. For the calculation of topological correlators, to regard \tilde{Q} as a BRST and to make the theory truly topological is a matter of preference. One can consider the physical theory and still calculate correlators in the topological sector. Simplest examples of this type is the supersymmetric quantum mechanics with discrete spectrum. (For continuous spectrum, supersymmetry does not imply the equality of density of states in the bosonic and fermionic sector and the following statements needs refinement.) For example, in the calculation of topological partition function, (Witten index), one can sum over all states in the Hilbert space. There is an exact cancellation between paired bosonic and fermionic states with nonzero energy, and hence only zero energy states contribute. Alternatively, one can declare the theory topological, and the physical states (Q -cohomology group) of the topological theory are just the quantum ground states of the full theory. The rest of the Hilbert space is redundant in the sense of BRST, and the partition function only receives contribution from quantum ground states. In the language of path integrals, this translates to localization of appro-

privately defined correlators to the the fixed point of the Q -action in the supersymmetry transformations, such as the ones eq. (3.32) and eq. (3.35). Therefore, there are stronger techniques to calculate topological correlators. See for example for a review [38].

Marcus shows that the fixed point of eq. (3.32) is the space of complexified flat connections. He also argues that the theory reduces to Donaldson-Witten theory if one demands a reality condition and use the \tilde{Q} in eq. (3.35) [6].¹³ Then, in eq. (3.35), both field strengths reduce to the usual field strength, i.e, $\overline{\mathcal{F}}_{\mu\nu} = \mathcal{F}_{\mu\nu} = F_{\mu\nu}$. The vanishing condition of the final equation in that case becomes the instanton equation $F_{\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} = 0$ and the path integral can be expanded around the instantons. Here, we do not wish to make such an assumption and just consider the theory as it is. This gives a complex version of instanton equations which relates the holomorphic field strength to the dual of the anti-holomorphic field strength:

$$\overline{\mathcal{F}}^{(2)} + *\mathcal{F}^{(2)} = 0, \quad \overline{\mathcal{F}}_{\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\rho\sigma} = 0 \tag{3.37}$$

or equivalently using eq. (3.4), we can split it into its hermitian and antihermitian parts. In this case, the equation takes the form:

$$\begin{aligned} F_{\mu\nu} - i[S_\mu, S_\nu] + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}(F^{\rho\sigma} - i[S^\rho, S^\sigma]) &= 0 \\ (D_\mu S_\nu - D_\nu S_\mu) - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}(D^\rho S^\sigma - D^\sigma S^\rho) &= 0 \end{aligned} \tag{3.38}$$

I do not know the full set of solutions to these equations. However, it seems rather plausible that the moduli space (as in Donaldson-Witten theory) is just isolated instantons under circumstances analyzed in [1]. Then, by using the weak coupling limit of the theory and by exploiting the coupling constant independence of the partition function, one can calculate certain observables. It seems sufficient to keep the quadratic part of the Lagrangian (owing to weak coupling) and benefit from the steepest decent techniques. If all this holds, then the fermionic determinant around such instanton configurations should be real, and by supersymmetry should be related to the bosonic determinant. This simply follows from the equality of nonzero eigenvalues of the bosonic and fermionic quantum fluctuations around the instanton background. For example, under circumstances where the dimension of the instanton moduli space is zero, and hence there are no fermionic zero modes, the partition function of the theory should be a topological invariant [1] and should be calculable without any reference to fermion sign problem. Similar considerations also hold for other topological correlators. The main point is that the fixed points of some Q -actions may lead to the finite action field configurations which admits the saddle-point approximations. In the case where the observables are independent of coupling constants, the partition function localizes to these fixed points and hence dominates the path integral. Under such circumstances, one can evade fermion sign problem. Also see [38] about localization.

¹³This reality condition is not compatible with the gauge invariance on the lattice construction. If it were possible to implement this condition, this would yield a lattice formulation of $\mathcal{N} = 2$ SYM theory in four dimensions and would be remarkable.

4. The Blau-Thompson twists and three dimensional lattices

In this section, we show that the orbifold lattice action of the three dimensional theories produce the Blau-Thompson type twists [7]. In each case, we will see that the point group symmetry of the lattice action enhances to the twisted rotation group $SO(d)'$ in the continuum. We will also observe that a continuous R -symmetry which has the same rank as the R -symmetry of the continuum twisted theory is exactly realized on the cubic orbifold lattices. The features of these lattices in the sense of representation theory follows very similar pattern to our discussion in the previous section. Namely, there are always spacetime scalars in vector representation of the twisted rotation group and hence lattice, and the double valued spinor representation of the continuum theory are always associated with the single valued representations of the orbifold lattice theories. Since all the tools that we need to use are developed in the previous section, our presentation will be brief and will emphasize symmetries rather than technical details.

4.1 The $\mathcal{N} = 4$ SYM in $d = 3$

The $\mathcal{N} = 4$ SYM theory in three dimensions possess a global $G = SU(2)_E \times SU(2)_{R_1} \times SU(2)_{R_2}$ where $SU(2)_E \sim SO(3)_E$ is the Euclidean Lorentz symmetry and $SU(2)_{R_1} \times SU(2)_{R_2}$ is the R -symmetry of the theory. To construct the Blau-Thompson twist [7], we take the diagonal subgroup of the spacetime $SO(3) \sim SU(2)_E$ and $SU(2)_{R_1}$. The twisted theory possess an $SO(3)' \times SU(2)_{R_2}$ symmetry.

Under G , the vector boson, scalars and fermions transform as $(3, 1, 1)$, $(1, 3, 1)$, $(2, 2, 2)$. In the twisted theory, the gauge bosons and scalars are on the same footing and they transform as $(3, 1)$. The fermions splits as $(3, 2) \oplus (1, 2)$, both of which are doublets under $SU(2)_{R_2}$. However, our lattice only respects the $U(1)$ subgroup of the $SU(2)_{R_2}$ and the full $SU(2)_{R_2}$ only emerges in the continuum. Therefore, we will express the continuum action with manifest $G' = SO(3)' \times U(1)$ symmetry. The fermions and bosons under G' transform as

$$\text{fermions} \rightarrow 1_{\frac{1}{2}} \oplus 3_{-\frac{1}{2}} \oplus 3_{\frac{1}{2}} \oplus 1_{-\frac{1}{2}}, \quad \text{bosons} \rightarrow 3_0 \oplus 3_0.$$

We label the fermions as $(\lambda, \psi^\mu, \xi_{\mu\nu}, \xi^{\mu\nu\rho})$. The action of the twisted theory is

$$\begin{aligned} \mathcal{L} &= \frac{1}{g^2} \text{Tr} \left[Q \left(\lambda \left(\frac{1}{2} id + \frac{1}{2} [\bar{\mathcal{D}}_\mu, \mathcal{D}^\mu] \right) + \frac{i}{4} \xi_{\mu\nu} \mathcal{F}_{\mu\nu} \right) + \frac{1}{2} \xi^{\mu\nu\rho} \bar{\mathcal{D}}_\mu \xi_{\nu\rho} \right] \\ &= \frac{1}{g^2} \text{Tr} \left[\frac{1}{8} ([\bar{\mathcal{D}}_\mu, \mathcal{D}^\mu])^2 + \frac{1}{4} |\mathcal{F}_{\mu\nu}|^2 + \lambda \bar{\mathcal{D}}_\mu \psi^\mu + \xi_{\mu\nu} \mathcal{D}^\mu \psi^\nu + \frac{1}{2} \xi^{\mu\nu\rho} \bar{\mathcal{D}}_\mu \xi_{\nu\rho} \right] \end{aligned} \quad (4.1)$$

The off-shell Q -transformations are given by

$$\begin{aligned} Q\lambda &= -id, & Qd &= 0 \\ Qz^\mu &= \sqrt{2}i\psi^\mu, & Q\psi^\mu &= 0, \\ Q\bar{z}_\mu &= 0 & \mu &= 1, \dots, 3 \\ Q\xi_{\mu\nu} &= -i\bar{\mathcal{F}}_{\mu\nu} \\ Q\xi^{\mu\nu\rho} &= 0 \end{aligned} \quad (4.2)$$

where $Q^2 = 0$. The action is a sum of a Q -exact and Q -closed term. The Q -invariance of the Q -closed term may be seen by the use of Jacobi identity eq. (3.13).

Cubic Lattice: The three dimensional orbifold lattice action for $\mathcal{N} = 4$ $d = 3$ theory [12] is a simple latticization of the Blau-Thompson twist of the theory. The lattice possess an $S_3 \times Z_2$ point group and a continuous $U(1)$ R -symmetry group. Inspecting the fermionic degrees of freedom; we observe that the fermions are associated with p -cells: one site, three links, three faces and one cube.

$$\text{fermions} \rightarrow 1 \oplus 3 \oplus 3 \oplus 1. \quad (4.3)$$

In the continuum, they fill the antisymmetric tensor representation of $SO(3)'$. Similarly, the vector bosons reside on the links, and they distribute as

$$\text{bosons} \rightarrow 3 \oplus 3. \quad (4.4)$$

In the continuum, they form the vector presentation of $SO(3)'$. The lattice formulation along with the details of the superfield formulation of the twisted theory is given in [12].

4.2 The $\mathcal{N} = 8$ SYM in $d = 3$

The $\mathcal{N} = 8$ SYM in $d = 3$ theory in $d = 3$ dimensions possess a global $G = SO(3)_E \times SO(7)_R$ symmetry. Under G , the fields transform as

$$\text{fermions} \rightarrow (2, 8), \quad \text{gauge boson} \rightarrow (3, 1), \quad \text{scalars} \rightarrow (1, 7), \quad (4.5)$$

In order to construct the Blau-Thompson twist [7, 33] we decompose the R -symmetry as $SO(7)_R \rightarrow SO(3)_{R_1} \times SO(4)_{R_2} \sim SU(2)_{R_1} \times (SU(2) \times SU(2))_{R_2}$. Under this decomposition, the scalars and fermions splits as $7 \rightarrow (3, 1, 1) \oplus (1, 2, 2)$ and $8 \rightarrow (2, 2, 1) \oplus (2, 1, 2)$. As usual, we take the diagonal sum of the Euclidean rotation group and the R_1 -symmetry group.

The twisted theory is invariant under $G' = SO(3)' \times (SU(2) \times SU(2))_{R_2}$, and the fields transform under G' as

$$\begin{aligned} \text{fermions} &\rightarrow (1, 2, 1) \oplus (3, 2, 1) \oplus (3, 1, 2) \oplus (1, 1, 2) \\ \text{gauge boson} &\rightarrow (3, 1, 1), \quad \text{scalars} \rightarrow (3, 1, 1) \oplus (1, 2, 2). \end{aligned} \quad (4.6)$$

As in the case of the $\mathcal{N} = 4$ theory in three dimensions, even though $(SU(2) \times SU(2))_{R_2}$ is a symmetry of the continuum theory, the orbifold lattice only respects an abelian $U(1) \times U(1)$ subgroup. The full non-abelian symmetry emerges as an accidental symmetry in the continuum. The transformation of the fields under $SO(3)' \times U(1) \times U(1)$ may be summarized as

$$z \oplus \bar{z} \rightarrow 1_{1,0} \oplus 1_{-1,0}, \quad z^\mu \oplus \bar{z}_\mu \rightarrow 3_{0,0} \oplus 3_{0,0}, \quad z_{\mu\nu\rho} \oplus \bar{z}^{\mu\nu\rho} \rightarrow 1_{0,1} \oplus 1_{0,-1} \quad (4.7)$$

for the ten bosonic degree of freedom and

$$\begin{aligned} \lambda \oplus \psi^\mu \oplus \xi_{\mu\nu} \oplus \chi^{\mu\nu\rho} &\rightarrow 1_{\frac{1}{2},\frac{1}{2}} \oplus 3_{-\frac{1}{2},-\frac{1}{2}} \oplus 3_{\frac{1}{2},\frac{1}{2}} \oplus 1_{-\frac{1}{2},-\frac{1}{2}} \\ \alpha \oplus \chi_\mu \oplus \bar{\xi}^{\mu\nu} \oplus \psi_{\mu\nu\rho} &\rightarrow 1_{\frac{1}{2},-\frac{1}{2}} \oplus 3_{-\frac{1}{2},\frac{1}{2}} \oplus 3_{\frac{1}{2},-\frac{1}{2}} \oplus 1_{-\frac{1}{2},+\frac{1}{2}} \end{aligned} \quad (4.8)$$

for the sixteen fermions.

Cubic Lattice: The three dimensional lattice action for $\mathcal{N} = 8$ $d = 3$ theory reproduces the Thompson-Blau twist in the continuum. The symmetries of the cubic lattice action are $S_3 \times Z_2 \times Z_2$ point group and a $U(1) \times U(1)$ R -symmetry group. The distribution of the fermions on the lattice follows very similar pattern to the $\mathcal{N} = 4$ theory. Since the number of fermions is doubled with respect to $\mathcal{N} = 4$, each p -cell accommodates twice as many fermions. The fermions distributes to p -cells as

$$\text{fermions} \rightarrow 2(1 \oplus 3 \oplus 3 \oplus 1) \tag{4.9}$$

on the lattice. Similarly, the elementary bosons reside on the sites, links, and 3-cell and they distribute as

$$\text{bosons} \rightarrow 2(1 \oplus 3 \oplus 1) . \tag{4.10}$$

In the continuum, they are scalars, vectors and antisymmetric third rank tensors under $SO(3)'$.

A_3^* (bcc) lattice: In order to see the Blau-Thompson twist from the body centered cubic (bcc) lattice, we follow the strategy of section 3.3. The lattice action possess the octahedral symmetry $O_h \sim S_4 \times Z_2$ where S_4 is the permutation group and Z_2 is the inversion group. The lattice also has a charge conjugation symmetry.¹⁴

We classify fields under the rotational subgroup A_4 of the tetrahedral group S_4 .¹⁵ Therefore, we consider the group action from each of the four conjugacy classes of A_4 and calculate the characters. For the one index link fields, we find that there is an A_4 -invariant subspace as in the case of the four dimensional lattice and these link fields are indeed reducible. The two index link fields are also reducible. A simple calculation yields

$$\begin{aligned} \chi(\lambda) &= \chi(\alpha) = (1, 1, 1, 1) = \chi_1 \\ \chi(z^m) &= \chi(\psi^m) = \chi(\bar{z}_m) = \chi(\bar{\psi}_m) = (4, 1, 1, 0) = \chi_4 \oplus \chi_1 \\ \chi(\xi_{mn}) &= (6, 0, 0, -2) = \chi_4 \oplus \chi_4 , \end{aligned} \tag{4.11}$$

which is a natural counterpart of the result eq. (3.23).

As in the discussion of A_5 symmetry group, there is an analogous four times four matrix \mathcal{E} which splits all the $O(g) \in A_4$ into block-diagonal form. This matrix is used to identify the irreducible representations of the A_4 group with the ones of the twisted rotation group

¹⁴The octahedral symmetry group O_h may be constructed in two different ways. One is $O_h \sim T_d \times Z_2$ where T_d is the symmetry group of tetrahedron and the other is $O_h \sim O \times Z_2$ where O is the rotation subgroup of O_h . The Z_2 is inversion. In identifying the lattice fields with the Blau-Thompson twisted version of the continuum, it is sufficient to work with $S_4/Z_2 = A_4$. Both A_4 and S_4 respect holomorphy for bosonic multiplets. The S_4 group actions on lattice fields turns bosonic (anti)-chiral supermultiplets into (anti)-chiral ones. However, the fermionic chiral and anti-chiral multiplets mixes under the S_4 action. The Z_2 inversion exchanges chiral and antichiral bosonic multiplets.

¹⁵The same lattice structure also shows up in spatial lattice formulation of $d = 4$ dimensional $\mathcal{N} = 4$ theory which is suitable for a Hamiltonian formulation [10]. The analysis of the irreducible representations of the full $S_4 \times Z_2 \times Z_2$ symmetry (the last Z_2 is charge conjugation) should be helpful to map the correlation functions of the continuum to the ones on the lattice.

classes:	(1)	(123)	(132)	(12)(34)
sizes:	1	4	4	3
χ_1	1	1	1	1
χ_2	1	ω	ω^*	1
χ_3	1	ω^*	ω	1
χ_4	3	0	0	-1

Table 5: The character table of A_4 , the pure rotation subgroup of tetrahedron. The full point group symmetry of the lattice action on A_3^* lattice is $S_4 \times Z_2$

$SO(3)'$. Thus, in the continuum of A_3^* lattice, we identify $\mathcal{E}_{m\mu}z^m = z^\mu$, $\mathcal{E}_{m4}z^m = \epsilon_{\mu\nu\rho}z_{\mu\nu\rho}/6$ for the fields associated with links. Similarly, the two index fermions of the A_3^* lattice are identified with the continuum fermions as $\xi_{mn}\mathcal{E}_{m\mu}\mathcal{E}_{n\nu} = \xi_{\mu\nu}$ and $\xi_{mn}\mathcal{E}_{m\mu}\mathcal{E}_{n4} = \epsilon_{\mu\nu\rho}\xi^{\nu\rho}/2$. Further details, including a superfield formulation of the Blau-Thompson twist can easily be extracted from section four of ref. [8].

5. Two dimensional examples

5.1 A new twist of the $\mathcal{N} = (2, 2)$ SYM theory

The $\mathcal{N} = (2, 2)$ SYM in $d = 2$ can be obtained by dimensional reduction of four dimensional $\mathcal{N} = 1$ SYM theory down to two dimensions. The theory possess a global $G = SO(2)_E \times SO(2)_{R_1} \times U(1)_{R_2}$ symmetry where $SO(2)_E$ is Euclidean Lorentz symmetry, $SO(2)_{R_1}$ is the symmetry due to reduced dimensions and $U(1)_{R_2}$ is the R -symmetry of the theory prior to reduction. The twisted Lorentz group $SO(2)'$ is the diagonal subgroup of $SO(2)_E \times SO(2)_{R_1}$. The vector V_μ transforming as $(2, 1)_0$ and the scalar S_μ transforming as $(1, 2)_0$ under G become $(2)_0$ under $G' = SO(2)' \times U(1)_{R_2}$. We complexify these fields into z^μ and \bar{z}_μ as in eq. (3.2). To see the transformation properties of the fermions is a little bit tricky, since the fermions transform under the spin group of $SO(4)_E$, i.e, $SU(2) \times SU(2)$ (before the reduction). However, the reduction is inherently real, and splits $SO(4) \rightarrow SO(2)_E \times SO(2)_{R_1}$.

In order to understand the transformation properties of fermions, we will take advantage of the relation between bispinors and vectors in four dimension. Let v_a , $\bar{\omega}$, and ω be the gauge field, the left and right handed spinors of the $d = 4$ theory where $a = 1, \dots, 4$. They transform under $SU(2) \times SU(2) \times U(1)_R$ respectively as $(2, 2)_0$, $(2, 1)_{-\frac{1}{2}}$, $(1, 2)_{\frac{1}{2}}$. We can turn the vector into a bispinor by using $\bar{\sigma}_a = (1, i\vec{\sigma})$ where $\vec{\sigma}$ is Pauli matrices and 1 is the two dimensional identity matrix. ω_α ($\bar{\omega}_{\dot{\alpha}}$) carries an undotted (dotted) spinor index α ($\dot{\alpha}$) and the index structure of the sigma matrix is $(\bar{\sigma}_\mu)_{\dot{\alpha}\alpha}$. Now, we construct the bispinors $v_{\dot{\alpha}\alpha} = (v_a \bar{\sigma}_a)_{\dot{\alpha}\alpha}$ and $\bar{\omega}_{\dot{\alpha}} \omega_\alpha$. These two bispinor transform identically under $SU(2) \times SU(2) \times U(1)_R$ as $(2, 2)_0$. The $v_{\dot{\alpha}\alpha}$ can suitably be expressed in terms of complexified

SO(2)' doublets \bar{z}_μ and z^μ . We have

$$v_{\dot{\alpha}\alpha} = \sqrt{2} \begin{pmatrix} \bar{z}_1 & -z^2 \\ \bar{z}_2 & z^1 \end{pmatrix}, \quad \text{and} \quad \bar{\omega}_{\dot{\alpha}\alpha}\omega_\alpha = \begin{pmatrix} \bar{\omega}_1\omega_1 & \bar{\omega}_1\omega_2 \\ \bar{\omega}_2\omega_1 & \bar{\omega}_2\omega_2 \end{pmatrix} \quad (5.1)$$

where the columns are SO(2)' doublets, \bar{z}_μ and $\epsilon_{\mu\nu}z^\nu$. From eq. (5.1), we see that $\bar{\omega}_{\dot{\alpha}\alpha}\omega_1$ and $\bar{\omega}_{\dot{\alpha}\alpha}\omega_2$ has to be SO(2)' doublets (vectors). Comparing with the columns of $v_{\dot{\alpha}\alpha}$ matrix, we identify $\bar{\omega}_{\dot{\alpha}}$ with an SO(2)' vector, ω_1 with a scalar and $\epsilon_{\mu\nu}\omega_2$ with a second rank antisymmetric tensor. We label these accordingly as $\psi^\mu, \lambda, \xi_{\mu\nu}$. Therefore, under the twisted symmetry SO(2)' \times U(1)_{R₂}, we obtain the transformation properties of the fermions and bosons as

$$\lambda \oplus \psi^\mu \oplus \xi_{\mu\nu} \rightarrow 1_{\frac{1}{2}} \oplus 2_{-\frac{1}{2}} \oplus 1_{\frac{1}{2}}, \quad z^\mu \oplus \bar{z}_\mu \rightarrow 2_0 + 2_0. \quad (5.2)$$

This is indeed the two dimensional counterpart of the twist introduced by [6, 7].

The off-shell supersymmetry transformation generated by the nilpotent scalar supercharge is given by

$$\begin{aligned} Q\lambda &= -id, & Qd &= 0, \\ Qz^\mu &= \sqrt{2}\psi^\mu, & Q\psi^\mu &= 0, \\ Q\bar{z}_\mu &= 0, & \mu &= 1, 2 \\ Q\xi_{\mu\nu} &= -i\bar{\mathcal{F}}_{\mu\nu}, \end{aligned} \quad (5.3)$$

where d as usual is an auxiliary field introduced for the off-shell completion of the supersymmetry algebra $Q^2 = 0$. This is clearly a Blau-Thompson and Marcus type twist, discussed in sections 3 and 4.1. The action of the twisted theory is given by a Q -exact expression

$$\begin{aligned} \mathcal{L} &= \frac{1}{g^2} Q \text{Tr} \left[\lambda \left(\frac{1}{2} id + \frac{1}{2} [\bar{\mathcal{D}}_\mu, \mathcal{D}^\mu] \right) + \frac{i}{4} \xi_{\mu\nu} \mathcal{F}^{\mu\nu} \right] \\ &= \frac{1}{g^2} \text{Tr} \left[\frac{1}{8} ([\bar{\mathcal{D}}_\mu, \mathcal{D}^\mu])^2 + \frac{1}{4} |\mathcal{F}^{\mu\nu}|^2 + \lambda \bar{\mathcal{D}}_\mu \psi^\mu + \xi_{\mu\nu} \mathcal{D}^\mu \psi^\nu \right]. \end{aligned} \quad (5.4)$$

The SO(2)' \times U(1)_{R₂} symmetry is manifest. Unlike the three and four dimensional counterparts, the action does not have a Q -closed term and its Q -invariance is manifest. This theory can be made topological by regarding Q as a BRST. The study of the corresponding topological theory may be interesting.

Square Lattice: The two dimensional orbifold lattice action for $\mathcal{N} = (2, 2)$ theory yields the Blau-Thompson type twist in the continuum [9]. We observe that the fermions on the lattice are associated with one site, two links and one face on each unit cell of the lattice. In the continuum, they fill, respectively, the scalar, vector and second-rank antisymmetric tensor representation of SO(2)'. The complex bosons are associated with the links (in both orientations) and they transform as vectors under SO(2)'. The continuum U(1)_{R₂} symmetry of the twisted theory is an exact symmetry on the lattice.

5.2 The $\mathcal{N} = (4, 4)$ SYM in $d = 2$

The $\mathcal{N} = (4, 4)$ SYM in $d = 2$ can be obtained by dimensionally reducing the six dimensional $\mathcal{N} = 1$ SYM theory down to two dimensions. The theory possess a $\text{SO}(2)_E \times (\text{SU}(2) \times \text{SU}(2))_{R_1} \times \text{SU}(2)_{R_2}$ symmetry group. The R_1 symmetry is the internal symmetry due to reduction from six down to two dimensions and R_2 is the R -symmetry of the theory prior to reduction. The twisted theory possesses a $\text{SO}(2)' \times \text{U}(1)_{R_1} \times \text{SU}(2)_{R_2}$ symmetry group. The orbifold lattice only respects the $\text{U}(1)$ subgroup of the $\text{SU}(2)_{R_2}$ and therefore we will express the representations of the fields under $G' = \text{SO}(2)' \times \text{U}(1) \times \text{U}(1)$. The six bosonic fields transform under G' as

$$z \oplus \bar{z} \rightarrow 1_{1,0} \oplus 1_{-1,0}, \quad z^\mu \oplus \bar{z}_\mu \rightarrow 2_{0,0} \oplus 2_{0,0} \quad (5.5)$$

The eight fermion spits into two groups of four as

$$\begin{aligned} \lambda \oplus \psi^\mu \oplus \psi_{\mu\nu} &\rightarrow 1_{\frac{1}{2}, -\frac{1}{2}} \oplus 2_{-\frac{1}{2}, \frac{1}{2}} \oplus 1_{\frac{1}{2}, -\frac{1}{2}}, \\ \bar{\lambda} \oplus \bar{\psi}_\mu \oplus \bar{\psi}^{\mu\nu} &\rightarrow 1_{\frac{1}{2}, \frac{1}{2}} \oplus 2_{-\frac{1}{2}, -\frac{1}{2}} \oplus 1_{\frac{1}{2}, \frac{1}{2}} \end{aligned} \quad (5.6)$$

This twist is examined in detail in [34].

Square Lattice: The two dimensional orbifold lattice action for $\mathcal{N} = (4, 4)$ theory yields the Blau-Thompson type twist in the continuum. Having twice as many fermion with respect to $\mathcal{N} = (2, 2)$ theory, each p -cell on the lattice accommodates twice as many fermions (in opposite orientation). Besides discrete lattice symmetries, the lattice also possess a continuous $\text{U}(1) \times \text{U}(1)$ symmetry. In the continuum, these symmetries enhances to $\text{SO}(2)' \times \text{U}(1)_{R_1} \times \text{SU}(2)_{R_2}$ symmetry of the twisted theory. The superfield formulation of the twisted continuum and lattice theory is given in [12].

5.3 The $\mathcal{N} = (8, 8)$ SYM in $d = 2$

The $\mathcal{N} = (8, 8)$ SYM theory possess an $\text{SO}(2)_E \times \text{SO}(8)_R$ symmetry group. The global symmetry of the twisted theory is $G' = \text{SO}(2)' \times \text{SU}(2) \times \text{SU}(2) \times \text{U}(1)$. The ten bosons transform under G' as

$$\begin{aligned} z^\mu \oplus \bar{z}_\mu &\rightarrow (2, 1, 1)_0 \oplus (2, 1, 1)_0 \\ z_{\mu\nu} \oplus \bar{z}^{\mu\nu} &\rightarrow (1, 1, 1)_1 \oplus (1, 1, 1)_{-1}, \quad \tilde{z} \rightarrow (1, 2, 2)_0 \end{aligned} \quad (5.7)$$

For the sixteen fermionic degree of freedom, we obtain

$$\begin{aligned} \lambda \oplus \psi^\mu \oplus \psi_{\mu\nu} &\rightarrow (1, 2, 1)_{\frac{1}{2}} \oplus (2, 2, 1)_{-\frac{1}{2}} \oplus (1, 2, 1)_{\frac{1}{2}} \\ \bar{\lambda} \oplus \bar{\psi}^\mu \oplus \bar{\psi}^{\mu\nu} &\rightarrow (1, 1, 2)_{-\frac{1}{2}} \oplus (2, 1, 2)_{\frac{1}{2}} \oplus (1, 1, 2)_{-\frac{1}{2}} \end{aligned} \quad (5.8)$$

Square Lattice: The fermions are distributed in multiples of four to each p -cell as $4(1 \oplus 2 \oplus 1)$. Four of the bosons (labeled as \tilde{z}) are associated with site, four of them (z^μ and \bar{z}_μ) with the links each accommodating two, and two of them $z_{\mu\nu}$ and $\bar{z}^{\mu\nu}$ on the face diagonal. For details, see [8].

A_2^* (Hexagonal) Lattice: The A_2^* orbifold lattice action possess a point group symmetry $S_3 \times Z_2$, where S_3 is the permutations of the chiral multiplets and Z_2 is the inversion symmetry swapping chiral and antichiral multiplets. Another discrete symmetry of the action is charge conjugation. Following the analysis of the A_3^* and A_4^* lattices, it is sufficient to construct the pure rotation subgroup A_3 of S_3 to make connection to the twisted form. However, A_3 is an abelian cyclic group and it only possess one dimensional representations. This is not a problem. Recall that the two dimensional vector representation of $SO(2)'$ is also reducible when we regard it in its spin group, $U(1)'$. The A_3 character table has two complex conjugate characters $\chi_2 = (1, e^{2\pi i/3}, e^{-2\pi i/3})$ and $\chi_3 = (1, e^{-2\pi i/3}, e^{2\pi i/3})$. These two complex conjugate representation of the A_3 group has to be regarded as one two dimensional representation. The sum $\chi = \chi_2 + \chi_3 = (2, -1, -1)$ is a two dimensional real character and is irreducible over \mathbb{R} . Alternatively, we can also work with the full nonabelian point group symmetry of the lattice. The $S_3 \times Z_2$ group has two dimensional representations and a little bit more information than we need here. ¹⁶

As in the A_3^* and A_4^* lattices, there is a A_3 -invariant subspace of the link fields, and consequently, the link field splits into a singlet and a two dimensional representation. We obtain the characters as

$$\begin{aligned} \chi(z^m) &= \chi(\bar{z}_m) = (3, 0, 0) = \chi \oplus \chi_1 \\ \chi(z) &= \chi(\lambda) = \chi(\bar{\lambda}) = (1, 1, 1) = \chi_1 \end{aligned} \tag{5.9}$$

Therefore, the sixteen fermions and ten bosons splits as

$$\text{fermions} \rightarrow 4(\chi_1 \oplus \chi \oplus \chi_1) \tag{5.10}$$

$$\text{bosons} \rightarrow 4\chi_1 \oplus 2\chi_3 \oplus 2\chi_1 \tag{5.11}$$

as in the continuum twisted theory discussed above. There is also an analogous matrix \mathcal{E} which maps the irreducible representations of S_3 (or A_3) into the ones of twisted rotation group $SO(2)'$. The matrix \mathcal{E} brings the group actions of A_3 into a block diagonal form. Thus, in the continuum of A_2^* lattice, the fields associated with the links become vector and scalar representation of $SO(2)'$. Explicitly, we have $\mathcal{E}_{m\mu} z^m = z^\mu$, $\mathcal{E}_{m3} z^m = \epsilon_{\mu\nu} z_{\mu\nu}/2$ and similar mappings for other fields.

6. Conclusions and prospects

Certainly one of the most bizarre features of the orbifold lattices was associating spinless bosons of the continuum theory with the link fields which transform nontrivially on the lattice, and associating double valued spinor representation of the continuum with the single valued representations of the point group of the lattice [10, 9, 12, 8, 11]. Remarkably, the orbifold lattice in the continuum gave Lorentz invariant, highly supersymmetric theories with no or little fine tuning. This work hopefully demystifies the orbifold lattices by relating them to the twisted versions of supersymmetric theories. Many twisted theories arise, in

¹⁶The same lattice structure also emerges for the spatial lattice of $\mathcal{N} = 8$ theory in $d = 3$ dimensions. See the footnote. ¹⁵

the continuum, as a courtesy of the orbifold projection. These twisted versions are often worked in the context of topological field theory, and we hope this work leads to further, fruitful interplay between these two branches. Before moving to the prospects, let us give the summary of our results:

- The orbifold lattices, in the continuum, reproduce the Marcus and Blau-Thompson twists of the extended supersymmetric theories. Conversely, it is possible to discretize (with a well-defined recipe) the Marcus and Blau-Thompson twists of the extended supersymmetric theories to obtain the orbifold lattice action.
- The point group symmetry of the orbifold lattice is a subgroup of the twisted Lorentz group, and not the real Lorentz group.
- The exact supersymmetries on the orbifold lattices are the nilpotent spin zero, scalar supersymmetries of the continuum twisted theory.
- The p -form fields on the continuum are naturally associated with p -cells on the hypercubic lattices. For more symmetric A_d^* lattices, the irreducible representations of lattice rotation group are in one to one correspondence with the representations of twisted rotation group.

It is also possible to understand the spatial orbifold lattices [10] and deconstruction of higher dimensional supersymmetric theories [39] from the viewpoint of the present work. They correspond to latticization of partial or half twisted versions of the corresponding target field theories. Also, a few new partial twisting of $\mathcal{N} = 2$, and $\mathcal{N} = 4$ in $d = 4$ supersymmetric Yang-Mills theory seems to exist.

It is clear that the twisted versions of the supersymmetric theories are in a more peaceful existence with lattice. The main point is that in the twisted theories some of the supercharges are spin zero scalars, and they do not make any reference to the underlying structure of spacetime. Even when carried into the lattice, the supersymmetry algebra $Q^2 = 0$ still holds with no reference to finite lattice translations. We believe this relation is the key for the lattice regularization for a larger class of supersymmetric theories. It seems that twisted versions of certain sigma-models in two dimensions may provide good opportunities. Some theories of this type are known to have an isolated, discrete vacua, a discrete spectrum and mass gap.

There are also interesting directions to explore in the continuum twisted versions. An interesting class of theories arises from the $SO(4)' \times U(1)$ and $\mathcal{Q} = 1$ symmetry preserving deformations of the twisted action eq. (3.17). Clearly, there is a few parameter family of deformations of eq. (3.17) satisfying these requirements. For example, altering \mathcal{L}_2 into $\mathcal{L}_2 = Q \text{Tr} \left(c_1 \frac{i}{4} \xi_{\mu\nu} \mathcal{F}^{\mu\nu} + c_2 \frac{1}{12\sqrt{2}} \xi^{\nu\rho\sigma} \mathcal{D}^\mu z_{\mu\nu\rho\sigma} \right)$, where c_1 and c_2 are deformation parameters is of this type. For \mathcal{L}_3 , the two $SO(4)' \times U(1)$ singlets are glued to each other because of $\mathcal{Q} = 1$ supersymmetry, however an overall parameter is possible. Only for a special choice of the deformation parameters (for example $c_1 = c_2 = 1$ etc), and in flat spacetime, this theory eq. (3.17) is a rewriting of $\mathcal{N} = 4$ SYM and is under the strong protection of underlying higher symmetry, sixteen supersymmetries. The other theories, for example

with $c_1 \neq c_2$ may be worth exploring, both in flat and curved spacetimes. The most natural framework to think about such deformations seems to be (Euclidean) D3-branes wrapped on curved four manifolds. It is well-known that the world-volume of the wrapped D-branes do not realize the usual form of the supersymmetry, but a twisted version of it. The constructions in this paper can be considered as a straightforward realization of this idea, because underlying manifold (in continuum) is flat, d -dimensional torus T^d .

Another issue which arises from the twisted versions are related to BPS solitons. As in the Witten's treatment of Donaldson theory [1], where instantons appears as fixed points of supersymmetry transformations, the vanishing of fields under \tilde{Q} in eq. (3.35) produce a complex generalization of the instanton equation. Similar considerations for the Blau-Thompson twists of $d = 3$, $\mathcal{N} = 8$ theory yields a complex generalization of monopole equation. It is desirable to understand these solitons in more detail, and in particular the structure of their moduli spaces. Research in this direction is ongoing.

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A. Twistings by discrete R -symmetries and finite spin groups

In this appendix, I will briefly sketch an alternative view on twisting, from the viewpoint of discrete groups. For clarity, I will distinguish the groups with double valued representation from the ones with single valued representations. For example, spin groups will be treated differently from the orthogonal group.

In this paper, we considered theories with sufficiently large R -symmetries such that a nontrivial homomorphism from the full Lorentz group to the R -symmetry group was possible. We performed twists of a rather simple kind by constructing the diagonal sum

$$\text{Diag}(Spin(d) \times Spin(d)) = Spin(d)' \quad (\text{A.1})$$

At the end, only integer spin representations appeared in $Spin(d)'$. These representations are p -form fields and are the shared representations with $SO(d)'$. That means, in the twisted theory, we really do not need to think of spin group anymore since there are no spinor representations at all. One of the main observation of this paper is that the point group symmetry of the supersymmetric orbifold lattices is a finite subgroup of the $SO(d)'$.

Can we understand the above construction in the language of finite groups? The answer is positive and complementary to the approach in the bulk of this paper. The answer clearly requires the knowledge of finite subgroups of spin groups. The classification of these groups is well know, and these are the spin groups of the point group symmetries.

Let us consider a particular case: The spin group of $SO(3)$ is $Spin(3) = SU(2)$. It is related to $SO(3)$ by a two to one map $SU(2)/\pm 1 = SO(3)$. Let us call this map π . We have $\pi : SU(2) \rightarrow SO(3)$. Given any finite subgroup G_f of $SO(3)$, we can look for $\pi^{-1}(G_f)$. This gives a list of finite subgroups of the $SU(2)$, which we label as \tilde{G}_f . Examples of \tilde{G}_f are $\tilde{A}_3, \tilde{A}_4, \tilde{S}_3, \tilde{S}_4$. These are respectively, the spin groups (doubling) of the finite groups A_3, A_4, S_3, S_4 which frequently appeared as the point group symmetry of the lattices. The doubled-groups \tilde{G}_f admit spinor representations. The number of conjugacy classes (hence characters) of \tilde{G}_f is always larger than the one of G_f , but usually not twice as much.

Let us consider a finite subgroup of $Spin(d)_L \times Spin(d)_R$, which we will label as $\tilde{G}_f \times \tilde{G}_f$. The first one of these corresponds to spacetime and the latter corresponds to R -symmetry. Let us assume the spacetime is discretized. Then the fields transforming in irreducible representations of $Spin(d)$ will split into irreducible representations of \tilde{G}_f . For low dimensional representation of $Spin(d)$, there is usually a single corresponding representation in \tilde{G}_f and there is no splitting. Of course, high dimensional representations of the $Spin(d)$ will split into many representation of \tilde{G}_f since, simply, the representations of \tilde{G}_f are finite dimensional. This is similar to the level splitting of an atom inserted into a field of crystal potential which has a finite symmetry group. Assuming the perturbing potential has a lower symmetry, the degeneracies are determined by the representations of the perturbation. In our case, for the fields appearing in Lagrangian, there is usually just a single representation to be matched with in lattice. In order to obtain the orbifold lattices, it seems inevitable that the internal R -symmetry has to be restricted to a finite spin group as well. This finite R -spin group has to be necessarily identical to the \tilde{G}_f of spacetime for the desired outcome.

Let us consider an example: A spacetime spinor fermion $\psi_{\dot{\alpha},\alpha}$ in the bi-spinor representation of $Spin(3)_L \times Spin(3)_R$. It transforms under $Spin(3)_L \times Spin(3)_R$ as $\psi \rightarrow L\psi R^\dagger$ with obvious action of L and R . Let us assume that L is an element of double-group \tilde{G}_f , and let us consider a particular combination of the field such as $\text{Tr } \psi$. (I will come back to other components momentarily.) Then, it is clear that whatever L action we choose, the field $\text{Tr } \psi$ will remain invariant as long as I restrict R to discrete operations $R = L$. In that case $\text{Tr } \psi \rightarrow \text{Tr } L\psi R^\dagger = \text{Tr } L\psi L^\dagger = \text{Tr } \psi$. That means the field $\text{Tr } \psi$ is invariant under the diagonal sum of $\tilde{G}_f \times \tilde{G}_f$. Let us call this diagonal subgroup \tilde{G}'_f . Then we can define the twisted discrete point group as

$$\text{Diag}(\tilde{G}_f \times \tilde{G}_f) = \tilde{G}'_f. \tag{A.2}$$

As in the case of its continuous counterpart, there are no double-valued representations appearing in \tilde{G}'_f . Therefore, the group is really just G'_f , which is a subgroup of the twisted rotation group $SO(d)'$.

Now, let us come back to the other components of the bispinor field and treat them slightly more rigorously. The spin group \tilde{A}_4 has seven conjugacy classes (see, for example, [40], page 393) whereas as shown in table 5, the A_4 has only four. Rather than examining the details of representation of \tilde{A}_4 , we want to use necessary information to see the fate of the other components of $\psi_{\dot{\alpha},\alpha}$ field. The conjugacy classes with their multiplicities

are

$$(1), S, 4(123), 4(132), 4(123)S, 4(132)S, [3(12)(34) + 3(12)(34)S]$$

where S is the 2π rotation such that $S^2 = 1$. The character for the two dimensional spinor representation is $\chi(\psi_{\dot{\alpha}}) = (2, -2, 1, -1, -1, 1, 0)$. Under the diagonal \tilde{A}'_4 , $\psi_{\dot{\alpha},\alpha}$ transform as $\chi(\psi_{\dot{\alpha},\alpha}) = \chi(\psi_{\dot{\alpha}}) \times \chi(\psi_{\alpha}) = (4, 4, 1, 1, 1, 1, 0)$. The product splits into two representations, and these are indeed common representations with A'_4 . Therefore, it is sufficient to inspect the character table of A_4 . We conclude $\chi(\psi_{\dot{\alpha},\alpha}) = \chi_4 \oplus \chi_1$ as we expect. The bispinor $\psi_{\dot{\alpha},\alpha}$ splits into single valued, a three dimensional vector representation and a one dimensional scalar representation under A'_4 . Explicitly, we have

$$\psi_{\dot{\alpha},\alpha} = (\psi^0 1_2 + \psi_{\mu} \sigma^{\mu})_{\dot{\alpha},\alpha}, \quad \text{or } \psi^0 = \frac{1}{2} \text{Tr } \psi, \quad \psi_{\mu} = \frac{1}{2} \text{Tr } \psi \sigma_{\mu} \quad (\text{A.3})$$

where σ_{μ} are the usual Pauli matrices.

Finally, I do not know a lattice formulation which is supersymmetric and invariant under $\tilde{G}_f \times \tilde{G}_f$ or $\tilde{G}_f \times$ (full R -symmetry). The difficulty is that; under the real spacetime symmetry group scalars, gauge bosons and fermions are treated on very different footing on the lattice. However, the twisted version happily accommodates all while preserving a subset of supersymmetry.

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